HW #4 solutions (2025)

(1.) Note that $p_1 = p_2$ at thermal equilibrium.

(a) $T=0$

Degenerate Fermi gas: $\mathcal{U} = \frac{3}{5} N \epsilon_F$,

where $$\epsilon_F = \frac{\hbar^2}{2m}\left(\frac{6\pi^2}{(2s+1)}\, n\right)^{2/3}.$$

$$n = \frac{N}{V} = \frac{1}{v} \quad \text{particle density}$$

Then $$p = -\left(\frac{\partial \mathcal{U}}{\partial V}\right)_{\underset{\text{entropy}}{S}} = \underset{\text{const}(V,\, s)}{A}\, \frac{n^{5/3}}{(2s+1)^{2/3}}$$

(s ← spin)

Finally, $p_1 = p_2$ yields

$$\left(\frac{n_1}{n_2}\right)^{5/3} = \frac{(2s_1+1)^{2/3}}{(2s_2+1)^{2/3}}, \text{ or}$$

$$\frac{V_2}{V_1} = \frac{(2s_1+1)^{2/5}}{(2s_2+1)^{2/5}}.$$

(b) $T=\infty$

Classical ideal gas: $\frac{n_1}{n_2} = \frac{V_2}{V_1} = 1.$

$p = n k_B T$

(2.) 2D ideal Bose gas

(a) $\log \Xi = -\sum_{\vec{p}} \log(1 - \underbrace{z}_{\overset{\shortparallel}{e^{\beta\mu}}} e^{-\beta\epsilon_{\vec{p}}})$ $\circledequal$

$e^{\beta\mu}$, fugacity

$\circledequal -\frac{2\pi L^2}{h^2}\int_0^\infty dp\, p\, \underbrace{\log(1 - z e^{-\beta\epsilon_{\vec{p}}})}_{\text{expand in powers of } z}$, where

$\epsilon_{\vec{p}} = \frac{p^2}{2m}$; $\underbrace{L^2 = A}_{\text{area}}$

Furthermore,

$$\log \Xi = +\frac{2\pi L^2}{h^2}\sum_{k=1}^{\infty}\frac{1}{k}\int_0^\infty dp\, p \left(z e^{-\frac{\beta p^2}{2m}}\right)^k \circledequal$$

$$-\log(1-x) = \sum_{k=1}^{\infty}\frac{x^k}{k}$$

$$\circledequal \frac{2\pi L^2}{h^2}\sum_{k=1}^{\infty}\frac{z^k}{k}\underbrace{\int_0^\infty dp\, p\, e^{-\frac{k\beta p^2}{2m}}}_{\frac{m}{k\beta}} = \frac{2\pi L^2}{h^2}\frac{m}{\beta}\underbrace{\sum_{k=1}^{\infty}\frac{z^k}{k^2}}_{g_2(z)} \Leftrightarrow$$

$$\Leftrightarrow \frac{L^2}{2\pi\hbar^2} m k_B T\, g_2(z).$$

(b) Recall that

$$\frac{p}{k_B T} = \frac{\log \Sigma}{L^2}\,, \quad \text{then}$$

$$\frac{p}{k_B T} = \underbrace{\frac{m k_B T}{2\pi \hbar^2}}_{\lambda^{-2}} g_2(z) = \frac{g_2(z)}{\lambda^2}.$$

(c) $$\underset{\substack{\uparrow \\ \text{part. density}}}{n} = \frac{N}{L^2} = z\frac{\partial}{\partial z}\left(\frac{\log \Sigma}{L^2}\right) = z\frac{\partial}{\partial z}\left(\frac{g_2(z)}{\lambda^2}\right) =$$

$$= \lambda^{-2} z \sum_{k=1}^{\infty} \frac{z^{k-1}}{k} = \lambda^{-2} \underbrace{\sum_{k=1}^{\infty} \frac{z^k}{k}}_{-\log(1-z)} = -\frac{\log(1-z)}{\lambda^2}.$$

Finally, $$\log(1-z) = -n\lambda^2,$$

$$z = 1 - e^{-\lambda^2 n}, \quad \text{or}$$

$$\underset{\substack{\uparrow \\ \mu(n,T)}}{\mu} = k_B T \log\left(1 - e^{-\lambda^2 n}\right).$$