HW # 3 solutions (2025)

(1.) Elastic energy $\mathcal{U} = \frac{\kappa \varphi^2}{2}$, where φ is the angular deviation from equilibrium.

Then $$Z = \underbrace{\int_{-\infty}^{\infty}}_{\text{extend the limits since } \langle \varphi^2 \rangle \ll 1} d\varphi\, e^{-\underbrace{\frac{\kappa \varphi^2}{2 k_B T}}_{\alpha = \frac{\kappa}{2 k_B T}}} = \sqrt{2\pi}\,\sigma = \sqrt{\frac{\pi}{\alpha}}.$$

$$\sigma^2 = \frac{k_B T}{\kappa} = \frac{1}{\sqrt{2\alpha}}$$

Next, $$\langle \varphi^2 \rangle = -\frac{\partial}{\partial \alpha} \log Z = -\left(-\frac{1}{2}\right)\frac{1}{\alpha} = \frac{1}{2\alpha}.$$

$$\langle \varphi^2 \rangle = \frac{1}{2}\,\frac{2 k_B T}{\kappa} \Rightarrow \kappa \langle \varphi^2 \rangle = \underline{\underline{k_B T}}.$$

plugging in numbers,

$$k_B = \frac{9.428 \times 10^{-16}\ \frac{kg \cdot m^2}{s^2} \times 4.178 \times 10^{-6}\ rad^2}{287.1\ K} =$$

$$= 1.372 \times 10^{-23}\ \frac{J}{K}.$$

Using $k_B^{modern} = 1.381 \times 10^{-23}\ \frac{J}{K}$, we get

$\epsilon = 0.65\,\%$ error by magnitude.

(2.) Use $\mathcal{U} = 3pV$:

$$d\mathcal{U} = 3p\,dV + 3V\,dp = T\,dS - p\,dV.$$

Thus, $T\,dS = 4p\,dV + 3V\,dp$, yielding

$$\begin{cases} \left(\frac{\partial S}{\partial V}\right)_p = \frac{4p}{T}, \\ \left(\frac{\partial S}{\partial p}\right)_V = \frac{3V}{T}. \end{cases}$$

Now $\left(\frac{\partial}{\partial p}\left(\frac{\partial S}{\partial V}\right)_p\right)_V = \underbrace{4\left(\frac{\partial(p/T)}{\partial p}\right)_V = 3\left(\frac{\partial(V/T)}{\partial V}\right)_p}$.

Recall that $\underbrace{p}_{\text{intensive}} = \frac{\mathcal{U}}{\underbrace{3V}_{\frac{\mathcal{U}}{V} = \text{intensive}}} = p(T)$ only, cannot depend on V

Then $\frac{4}{T} - \frac{4p}{T^2}\frac{dT}{dp} = \frac{3}{T}$, ← $T = \text{const}$ if $p = \text{const}$

$$\frac{4p}{T}\frac{dT}{dp} = 1, \text{ or}$$

$$\underline{\underline{\frac{dp}{dT} = \frac{4p}{T}}} \quad (*)$$

$$p(T) = C_1 T^4 + \underbrace{C_2}_{=0 \text{ since } \lim_{T\to 0} p(T) = 0}.$$

← also, $\boxed{C_2 = 0}$ guaranteed by Eq. (*)

Finally, $\frac{\mathcal{U}}{V} = 3p \sim \underline{\underline{T^4}}$, as desired.

(3.) The allowed values of

$$\vec{k} = \frac{2\pi}{L}(n_1, \dots, n_d), \quad \text{where}$$

$$n_i = 0, \pm 1, \pm 2, \dots \qquad i = 1, \dots, d$$

Then $\epsilon(\vec{k}) = c|\vec{k}| = \frac{2\pi c}{L}\sqrt{\sum_{i=1}^{d} n_i^2}$.

(a) As before,

$$\log \Sigma = -\sum_{\vec{k}} \log\left(1 - e^{-\beta(\epsilon(\vec{k}) - \mu)}\right) =$$

$$= -\frac{\overset{\sim L^d}{V}}{(2\pi)^d} \underbrace{\int d^d\vec{k}}_{\int_0^\infty dk\, k^{d-1} \underbrace{S_d}_{\text{area of the unit sphere in } d \text{ dimensions}}} \log\left(1 - e^{-\beta(\epsilon(k) - \mu)}\right).$$

(b) $$\langle N \rangle = \left(\frac{\partial \log \Sigma}{\partial(\beta\mu)}\right)_{V,\beta} = \sum_{\vec{k}} \frac{1}{e^{\beta(\epsilon(k) - \mu)} - 1}, \quad \text{where}$$

$$\sum_{\vec{k}} \Rightarrow \frac{V}{(2\pi)^d}\int d^d\vec{k} \Rightarrow \frac{V}{(2\pi)^d} S_d \int_0^\infty dk\, k^{d-1} \quad \text{as in (a).}$$

So, $$\langle N \rangle = \left(\frac{L}{2\pi}\right)^d S_d \int_0^\infty dk\, k^{d-1} \frac{1}{e^{\beta(ck - \mu)} - 1}.$$

(c) Consider

$$\langle N \rangle = \left(\frac{L}{2\pi}\right)^d S_d \left(\frac{k_B T}{c}\right)^d \underbrace{\int_0^\infty dx \, \frac{x^{d-1}}{z^{-1} e^{x} - 1}}_{\equiv I(z)} ,$$

where $z = e^{\beta\mu}$.

Thus, $\underbrace{\frac{\langle N \rangle}{V}}_{\text{average density of particles}} \sim I(z)$.

Note that $I(z) \leq I(1) = \int_0^\infty dx \, \frac{x^{d-1}}{e^{x} - 1}$.

$I(1)$ diverges at $d=1$ and converges for $d \geq 2$. Since $I(1)$ represents the upper bound for the density of excited particles, BE condensation can happen for $d = 2, 3, 4, \dots$ ('extra' particles will go into the condensate), but not for $d=1$ where $I(1) \to \infty$.