HW #2 Solutions (2025)

(1.) Ideal gas eq'n of state:

$$k_B T = \frac{p}{\rho}, \quad \text{where } \rho = \frac{N}{V} \text{ is the air density}$$

Then $$dT = \frac{1}{k_B}\left(\frac{dp}{\rho} - \frac{p\,d\rho}{\rho^2}\right).$$

From problem 2, HW#1 we have:

$$V dp = -\gamma p dV, \quad \text{or}$$

$$\frac{d\left(\frac{1}{\rho}\right)}{\frac{1}{\rho}} = -\frac{d\rho}{\rho} = -\frac{1}{\gamma}\frac{dp}{p},$$

$$\frac{d\rho}{\rho} = \frac{1}{\gamma}\frac{dp}{p}.$$

Thus, $$dT = \frac{1}{k_B}\left(\frac{dp}{\rho} - \frac{\cancel{p}}{\rho}\frac{1}{\gamma}\frac{dp}{\cancel{p}}\right) = \frac{1}{k_B}\left(1-\frac{1}{\gamma}\right)\frac{dp}{\rho}.$$

Stevin's law yields:

$$\frac{dp}{\rho} = -mg\,dz \implies \frac{dT}{dz} = -\frac{mg}{k_B}\left(1-\frac{1}{\gamma}\right),$$

as requested

[Temperature decreases linearly with height]

2. (a) For a single dipole,

$$\mathcal{L} = K - \mathcal{U} = \underbrace{\frac{I}{2}\dot{\theta}^2 + \frac{I}{2}\sin^2\theta\,\dot{\varphi}^2}_{K} + \underbrace{\mu\cos\theta\, B}_{-\mathcal{U}} \quad B = |\vec{B}|.$$

(b) Momenta:
$$\begin{cases} p_\theta = \dfrac{\partial \mathcal{L}}{\partial \dot{\theta}} = I\dot{\theta}, \\ p_\varphi = \dfrac{\partial \mathcal{L}}{\partial \dot{\varphi}} = I\sin^2\theta\,\dot{\varphi}. \end{cases}$$

The Hamiltonian is given by:

$$H = K + \mathcal{U} = \frac{p_\theta^2}{2I} + \frac{p_\varphi^2}{2I\sin^2\theta} - \mu\cos\theta\, B.$$

The canonical partition function is $Z = Z_1^N$, where Z_1 is the single-dipole partition function:

$$Z_1 = \frac{1}{h^2}\int dp_\theta\, dp_\varphi\, d\theta\, d\varphi\; e^{-\frac{H}{k_B T}} =$$

h^2: to make Z_1 dimensionless

$$= \frac{1}{h^2}\int d\theta\, d\varphi \underbrace{\int dp_\theta\, dp_\varphi\, e^{-\frac{p_\theta^2}{2Ik_BT}}\, e^{-\frac{p_\varphi^2}{2I\sin^2\theta\, k_BT}}}_{2\pi(Ik_BT)\sin\theta}\, e^{\frac{\mu\cos\theta B}{k_BT}} \quad (1)$$

$$= \frac{2\pi (I k_B T)}{h^2} \int d\theta\, d\varphi \sin\theta\, e^{\frac{\mu \cos\theta B}{k_B T}}.$$

Using $B(\lambda) = \int d\theta\, d\varphi \sin\theta\, e^{\lambda\cos\theta} = 4\pi \frac{\sinh\lambda}{\lambda}$,

we obtain: $\left(\lambda = \frac{\mu B}{k_B T}\right)$

$$Z_1 = \frac{2\pi (I k_B T)}{h^2} 4\pi \frac{\sinh\left(\frac{\mu B}{k_B T}\right)}{\frac{\mu B}{k_B T}} =$$

$$= \frac{8\pi^2 I (k_B T)^2}{\mu B h^2} \sinh\left(\frac{\mu B}{k_B T}\right).$$

(c) $$m(B) = k_B T \frac{\partial \log Z_1}{\partial B} =$$

$$= k_B T \frac{\partial}{\partial B}\left[\log\left(\frac{1}{B}\right) + \log\sinh\left(\frac{\mu B}{k_B T}\right) + \text{const}(B)\right] =$$

$$= k_B T \left[-\frac{B}{B^2} + \frac{\cosh\lambda}{\sinh\lambda} \frac{\mu}{k_B T}\right] =$$

$$= \frac{\mu}{\tanh\lambda} - \frac{\mu}{\lambda}.$$

(3.) Grand canonical partition function:

$$\Sigma = \sum_{\nu} e^{-\beta E_\nu + \underbrace{\beta\mu}_{\alpha} N_\nu}.$$

Then $$\langle E \rangle = \frac{1}{\Sigma} \sum_{\nu} E_\nu e^{-\beta E_\nu + \underbrace{\beta\mu}_{\alpha} N_\nu} =$$
$$= -\left(\frac{\partial \log \Sigma}{\partial \beta}\right)_{\alpha},$$

$$\langle N \rangle = \frac{1}{\Sigma} \sum_{\nu} N_\nu e^{-\beta E_\nu + \beta\mu N_\nu} =$$

$$= \left(\frac{\partial \log \Sigma}{\partial (\beta\mu)}\right)_{\beta}.$$

Likewise,

$$\left(\frac{\partial^2 \log \Sigma}{\partial \beta^2}\right)_{\underbrace{\beta\mu}_{\alpha}} = -\left(\frac{\partial \langle E \rangle}{\partial \beta}\right)_{\beta\mu} =$$
$$= \langle E^2 \rangle - \langle E \rangle^2 = \langle \Delta E^2 \rangle,$$

$$\left(\frac{\partial^2 \log \Sigma}{\partial \alpha^2}\right)_{\beta} = \left(\frac{\partial \langle N \rangle}{\partial \alpha}\right)_{\beta} =$$
$$= \langle N^2 \rangle - \langle N \rangle^2 = \langle \Delta N^2 \rangle,$$

$$-\left(\frac{\partial}{\partial \beta}\left(\frac{\partial \log \Sigma}{\partial \alpha}\right)_{\beta}\right)_{\alpha} = -\left(\frac{\partial \langle N \rangle}{\partial \beta}\right)_{\alpha} = \langle EN \rangle - \langle E \rangle\langle N \rangle =$$
$$= \underline{\underline{\langle \Delta E \Delta N \rangle}}.$$