HW # 1 solutions (2025)

(1.) $S = k(NVE)^{1/3}$, yielding

$$\left(\frac{\partial S}{\partial E}\right)_{V,N} = \frac{k}{3}\left(\frac{NV}{E^2}\right)^{1/3}.$$

at equilibrium,

$$\frac{N^A V^A}{(E^A)^2} = \frac{N^B V^B}{(E^B)^2}, \text{ where}$$

$E^A + E^B = E = \text{const}.$

Thus, $(E - E^A)^2 N^A V^A = (E^A)^2 N^B V^B$,

$$\left(\frac{E^A}{E - E_A}\right)^2 = \frac{N^A V^A}{N^B V^B}, \text{ or}$$

$$\begin{cases} E^A = \dfrac{E_0}{1 + \sqrt{\dfrac{N^B V^B}{N^A V^A}}}, \\[2ex] E^B = \dfrac{E_0}{1 + \sqrt{\dfrac{N^A V^A}{N^B V^B}}}. \end{cases}$$

2.

(a) at $p = \text{const}$,

$$dS = C_p \frac{dT}{T}.$$

Then $dE = \underbrace{C_p dT}_{TdS} - p dV = (C_p - Nk_B) dT$

$p dV = N k_B dT$ for ideal gas @ $p = \text{const}$

On the other hand, $E = E(T)$ for an ideal gas:

$$dE = \underbrace{C_v}_{\left(\frac{\partial E}{\partial T}\right)_V} dT$$

← $dE = dQ$ since $V = \text{const}$, no mechanical work is done on the system

can also show formally that $\left(\frac{\partial E}{\partial V}\right)_T = 0$

Thus, $C_v = C_p - N k_B \Rightarrow \underline{\underline{C_p - C_v = N k_B}}$.

(b) Now we consider the adiabatic process: $dS = 0$.

We have $dE = C_v dT = - p dV$.

The equation of state now gives

$$dT = \frac{1}{Nk_B} d(pV) = \frac{1}{C_p - C_v} (p dV + V dp), \text{ or}$$

$$p dV + V dp = - \frac{C_p - C_v}{C_v} p dV,$$

$$V dp = - \frac{C_p}{C_v} p dV.$$

Finally, $\left(\frac{\partial p}{\partial V}\right)_S = - \frac{C_p}{C_v} \frac{p}{V} = - \gamma \frac{p}{V}$, as desired

(c) Use the equation of state again:

$$V = \frac{N k_B T}{p} \quad \text{gives}$$

$$dV = N k_B \left(\frac{dT}{p} - \frac{T}{p^2} dp \right) .$$

Using $V dp = -\gamma p dV$, we obtain:

$$\gamma \cancel{N} k_B T \frac{dp}{p} = \gamma \cancel{N} k_B dT + \frac{\cancel{N} k_B T}{p} dp \text{ , or}$$

$$\frac{dp}{p}(1-\gamma) + \gamma \frac{dT}{T} = 0 .$$

Since $\gamma = \text{const}$,

$$d \log \left(p T^{\gamma/(1-\gamma)} \right) = 0 \text{ , or}$$

$$p T^{\frac{\gamma}{1-\gamma}} = \text{const} \quad \text{for an adiabatic process}$$

3. $\gamma = \frac{C_p}{C_v}$, where

$$C_p = T\left(\frac{\partial S}{\partial T}\right)_p , \qquad C_v = T\left(\frac{\partial S}{\partial T}\right)_v .$$

Using
$$\begin{cases} \left(\frac{\partial S}{\partial T}\right)_p \left(\frac{\partial T}{\partial p}\right)_S \left(\frac{\partial p}{\partial S}\right)_T = -1, \\ \left(\frac{\partial S}{\partial T}\right)_v \left(\frac{\partial T}{\partial V}\right)_S \left(\frac{\partial V}{\partial S}\right)_T = -1, \end{cases}$$

we obtain:

$$\gamma = \frac{\left(\frac{\partial T}{\partial V}\right)_S \left(\frac{\partial V}{\partial S}\right)_T}{\left(\frac{\partial T}{\partial p}\right)_S \left(\frac{\partial p}{\partial S}\right)_T} \underset{\substack{\uparrow \\ \text{Maxwell} \\ \text{relations}}}{=} \frac{\left(\frac{\partial S}{\partial p}\right)_T \left(\frac{\partial V}{\partial S}\right)_T}{\left(\frac{\partial T}{\partial p}\right)_S \left(\frac{\partial V}{\partial T}\right)_S} =$$

$$= \frac{\left(\frac{\partial V}{\partial p}\right)_T}{\left(\frac{\partial V}{\partial p}\right)_S} = \frac{K_T}{K_S} , \quad \text{where}$$

$$\begin{cases} K_T = -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_T , \\ K_S = -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_S . \end{cases}$$

In more detail, recall that

$$dE = \underbrace{T}_{\left(\frac{\partial E}{\partial S}\right)_V} dS - \underbrace{p}_{-\left(\frac{\partial E}{\partial V}\right)_S} dV \quad \Rightarrow \quad \left(\frac{\partial T}{\partial V}\right)_S = -\left(\frac{\partial p}{\partial S}\right)_V \qquad (1)$$

$$dH = \underset{\substack{\shortparallel \\ \left(\frac{\partial H}{\partial S}\right)_p}}{T} dS + \underset{\substack{\shortparallel \\ \left(\frac{\partial H}{\partial p}\right)_S}}{V} dp \quad \Rightarrow \quad \left(\frac{\partial V}{\partial S}\right)_p = \left(\frac{\partial T}{\partial p}\right)_S \qquad (2)$$

Then

$$\gamma = \underset{\substack{\nearrow \\ (1)}}{-} \frac{\left(\frac{\partial p}{\partial S}\right)_V \left(\frac{\partial V}{\partial S}\right)_T}{\left(\frac{\partial T}{\partial p}\right)_S \left(\frac{\partial p}{\partial S}\right)_T} = \frac{\left(\frac{\partial V}{\partial S}\right)_T}{\left(\frac{\partial S}{\partial V}\right)_p \left(\frac{\partial V}{\partial p}\right)_S \left(\frac{\partial T}{\partial p}\right)_S \left(\frac{\partial p}{\partial S}\right)_T} =$$

$$\left(\frac{\partial p}{\partial S}\right)_V \left(\frac{\partial S}{\partial V}\right)_p \left(\frac{\partial V}{\partial p}\right)_S = -1$$

$$= \frac{\left(\frac{\partial S}{\partial p}\right)_T \left(\frac{\partial V}{\partial S}\right)_T}{\left(\frac{\partial T}{\partial p}\right)_S \underbrace{\left(\frac{\partial S}{\partial V}\right)_p \left(\frac{\partial V}{\partial p}\right)_S}} = \frac{\left(\frac{\partial S}{\partial p}\right)_T \left(\frac{\partial V}{\partial S}\right)_T}{\left(\frac{\partial T}{\partial p}\right)_S \underline{\underline{\left(\frac{\partial V}{\partial T}\right)_S}}}, \text{ as desired}$$

$$\left(\frac{\partial S}{\partial V}\right)_p \left(\frac{\partial V}{\partial T}\right)_S \left(\frac{\partial T}{\partial p}\right)_S \underset{\substack{\uparrow \\ (2)}}{=} \left(\frac{\partial V}{\partial T}\right)_S$$