Final solutions

(2025)

(1.) Choose a reference frame attached to the disk.

Consider particles hitting the disk from the right:

$$\underset{\substack{\uparrow \\ \text{particle} \\ \text{velocity}}}{\vec{w}} = (w_x, w_y, w_z) \quad , \quad \underbrace{w_x < 0}_{\substack{\text{condition for} \\ \text{a collision to} \\ \text{occur}}}$$

Momentum transfer: $\vec{\Delta p} = (2 m w_x, 0, 0)$.

The number of $\overbrace{\text{particles that}}^{\text{with velocities in } (\vec{w}, \vec{w}+d\vec{w}) \text{ interval}}$ hit the disk from the right per unit time is given by:

$$\Delta N(\vec{w}) = \rho \underbrace{P(\vec{w})}_{\substack{\text{prob. distribu-} \\ \text{tion of particle} \\ \text{velocities}}} |w_x| \underbrace{d\vec{w}}_{dw_x\, dw_y\, dw_z} \times \overbrace{S}^{= \pi R^2 \text{, the area of the disk}}$$

Here, $P(\vec{w}) = f(w_x + v) f(w_y) f(w_z)$,

with $f(w)$ given by the 1D Maxwell distribution:

$$f(w) = \sqrt{\frac{m}{2\pi k_B T}}\, e^{-\frac{m w^2}{2 k_B T}}$$

Note that $\langle w_x \rangle = -v$, as expected.

The total force acting on the disk due to collisions with gas particles on the right is given by:

$$F_x^R = -pS \times 2m \int_{-\infty}^{\infty} dw_y \int_{-\infty}^{\infty} dw_z \, f(w_y) f(w_z) \times$$

$$\times \int_{-\infty}^{0} dw_x \, w_x^2 f(w_x + v) =$$

$$= -2mpS \int_{-\infty}^{0} dw_x \, w_x^2 f(w_x + v) .$$

Likewise, the total force due to gas particles on the left (with $w_x > 0$) is given by:

$$F_x^L = 2mpS \int_{0}^{+\infty} dw_x \, w_x^2 f(w_x + v) .$$

Finally, the drag force is

$$F_x = F_x^R + F_x^L = 2mpS \Big[\int_{0}^{\infty} dw_x \, w_x^2 f(w_x + v) +$$

$$+ \underbrace{\int_{-\infty}^{0} dw_x \, w_x^2 f(w_x + v)}_{-\int_0^{+\infty} dw_x \, w_x^2 f(w_x - v)} \Big] \underset{v \text{ small}}{\approx} 4mpSv \underbrace{\int_0^{\infty} dw_x \, w_x^2 \frac{df}{dw_x}}_{\equiv I} .$$

Now, $$I \underset{\text{by parts}}{=} -2\int_0^\infty dv_x\, v_x f(v_x) =$$

$$= -2\sqrt{\frac{m}{2\pi k_B T}}\int_0^\infty dv_x\, v_x\, e^{-\frac{m v_x^2}{2k_BT}} \overset{u = v_x^2}{=}$$

$$= -\sqrt{\frac{m}{2\pi k_B T}}\underbrace{\int_0^\infty du\, e^{-\frac{mu}{2k_BT}}}_{\frac{2k_BT}{m}} = -\sqrt{\frac{2k_BT}{m\pi}}.$$

Thus, $$\boxed{\overset{\text{total drag force}}{F_x} = -4pSv\sqrt{\frac{2k_BTm}{\pi}}}.$$

2.

(a) $$\rho(E) = \underset{\substack{\uparrow \\ \#\text{ spin} \\ \text{states}}}{2}\,\frac{L^2}{(2\pi)^2}\int d^2k\;\delta\!\left(E - \frac{\hbar^2k^2}{2m}\right) =$$

$$= \frac{2L^2}{2\pi}\int_0^\infty dk\,k\;\delta\!\left(E - \frac{\hbar^2k^2}{2m}\right) \underset{u=k^2}{=} \frac{L^2}{2\pi}\int_0^\infty du\;\delta\!\left(E - \underbrace{\frac{\hbar^2u}{2m}}_{\equiv x}\right) =$$

$$= \underbrace{\frac{2m}{\hbar^2}\frac{L^2}{2\pi}}_{\frac{4\pi m L^2}{h^2}}\int_0^\infty dx\;\delta(E-x) = \Theta(E)\times\frac{4\pi m L^2}{h^2}.$$

$\uparrow$ constant for $E>0$

(b) $$N = \int_0^{\epsilon_F} dE\,\rho(E) = \frac{4\pi m L^2}{h^2}\,\epsilon_F,$$

yielding $$\boxed{\epsilon_F = \frac{h^2}{4\pi m}\,\overbrace{\rho}^{N/L^2}}.$$

(c) It can be shown that

$$\langle E\rangle = \frac{d}{2}\,P\,\underbrace{L^d}_{'V'},\quad \text{where } d = \#\text{ dims}.$$

Here, $d=2$: $$P = \frac{\langle E\rangle}{L^2} = \frac{1}{L^2}\int_0^{\epsilon_F} dE\,E\,\rho(E) =$$

$$= \frac{4\pi m}{h^2}\,\frac{\epsilon_F^2}{2} = \underline{\underline{\frac{h^2}{8\pi m}\,\rho^2}}.$$

(d) Use

$$N = \int_0^\infty dE \, \frac{\rho(E)}{e^{\beta(E-\mu)}+1} \quad \text{to find } \mu.$$

as $T \to 0$, $\quad N \to \underbrace{\frac{4\pi m L^2}{h^2}}_{\equiv \rho_0} \mu$.

On the other hand, $N = \rho_0 \epsilon_F$, s.t.

$\mu \to \epsilon_F$ in the $T \to 0$ limit.

(3.) (a) $\frac{df}{dm} = 0$ yields

$$m(a + bm^2 + cm^4) = h.$$

If $h=0$, $m=0$ is always a solution. Since $b>0$ and $c>0$, everything depends on the sign and magnitude of a.

→ If $a>0$, $f(m)$ has a minimum at $m=0$. ← order prm, symmetric phase in the

→ If $a<0$, $f(m)$ has a maximum at $m=0$ and two minima at $m = \pm m_0$ ← order prm, broken symmetry phase, where

$$m_0^2 = \frac{-b + \sqrt{b^2 - 4ac}}{2c}.$$

Clearly, $a=0$ is the critical point.

(b) In this case, $f(m)$ has a minimum at $m=0$ and another two minima at $m = \pm m_0$, where

$$a + bm_0^2 + cm_0^4 = 0. \qquad (1)$$

For phase coexistence, we require that $f(0) = f(\pm m_0)$, or

$$\frac{1}{2}a + \frac{1}{4}b\,m_0^2 + \frac{1}{6}c\,m_0^4 = 0 \,. \qquad (2)$$

Eqs. (1) and (2) together yield:

$$b\,m_0^2 + c\,m_0^4 = \frac{1}{2}b\,m_0^2 + \frac{1}{3}c\,m_0^4\,, \text{ or}$$

$$\underline{\underline{m_0^2 = -\frac{3b}{4c} > 0}}\,.$$

Finally, $a = -b\left(-\frac{3b}{4c}\right) - c\left(\frac{3b}{4c}\right)^2 =$

$$= \frac{3b^2}{4c} - \frac{9b^2}{16c} = \underline{\underline{\frac{3b^2}{16c} > 0}}\,.$$

(c) Since $a = b = 0$, $h = c\,m^5$,

yielding $\underline{\underline{\delta = 5}}$.