Transfer matrix techniques

Consider 1D Ising model: N spins

$$\mathcal{H} = -J \sum_{i=0}^{N-1} S_i S_{i+1} - H \sum_{i=0}^{N-1} S_i$$

Periodic BCs: $S_0 = S_N$, irrelevant in the $N \to \infty$ limit

Then

$$Z = \sum_{\{S\}} e^{\beta J (S_0 S_1 + S_1 S_2 + \ldots + S_{N-1} S_0) + \beta H (S_0 + S_1 + \ldots + S_{N-1})} =$$

$\uparrow$ sum over $S_i = \pm 1$, $\forall i$

$$= \sum_{\{S\}} e^{\beta J S_0 S_1 + \beta H \frac{S_0 + S_1}{2}} e^{\beta J S_1 S_2 + \beta H \frac{S_1 + S_2}{2}} \ldots$$

$$\ldots e^{\beta J S_{N-1} S_0 + \beta H \frac{S_{N-1} + S_0}{2}} =$$

Define $T^{i,i+1} = e^{\beta J S_i S_{i+1} + \beta H \frac{S_i + S_{i+1}}{2}}$

$$T = \begin{array}{cc} S_{i+1} = 1 & S_{i+1} = -1 \\ \end{array}$$
$$T = \left(\begin{array}{cc} e^{\beta(J+H)} & e^{-\beta J} \\ e^{-\beta J} & e^{+\beta(J-H)} \end{array} \right) \begin{array}{l} S_i = 1 \\ S_i = -1 \end{array}$$

Consider $\underline{N=2}$: $S_2 = S_0$

$$Z = \sum_{\{S\}} e^{\beta J S_0 S_1 + \beta H \frac{S_0 + S_1}{2}} e^{\beta J S_1 S_0 + \beta H \frac{S_1 + S_0}{2}} =$$

$\{S\} = \{S_0 = \pm 1, S_1 = \pm 1\}$

$$= e^{\beta J + \beta H} e^{\beta J + \beta H} + e^{\beta J - \beta H} e^{\beta J - \beta H} +$$

$S_0 = 1,\ S_1 = 1$ $\qquad$ $S_0 = -1,\ S_1 = -1$

$$+ \underbrace{e^{-\beta J} e^{-\beta J}}_{s_0 = 1,\ s_1 = -1} + \underbrace{e^{-\beta J} e^{-\beta J}}_{s_0 = -1,\ s_1 = 1} \quad \boxed{=}$$

On the other hand, consider

$$T^2 = \begin{pmatrix} e^{\beta(J+H)} & e^{-\beta J} \\ e^{-\beta J} & e^{+\beta(J-H)} \end{pmatrix} \begin{pmatrix} e^{\beta(J+H)} & e^{-\beta J} \\ e^{-\beta J} & e^{+\beta(J-H)} \end{pmatrix} =$$

$$= \begin{pmatrix} e^{\beta(J+H)} e^{\beta(J+H)} + e^{-\beta J} e^{-\beta J} & e^{\beta(J+H)} e^{-\beta J} + e^{+\beta(J-H)} e^{-\beta J} \\ e^{\beta(J+H)} e^{-\beta J} + e^{+\beta(J-H)} e^{-\beta J} & e^{-\beta J} e^{-\beta J} + e^{+\beta(J-H)} e^{+\beta(J-H)} \end{pmatrix}$$

$$\boxed{=} \ \mathrm{Tr}(T^2).$$

general case

$$\boxed{=} \sum_{\{s\}} T^{0,1} T^{1,2} \dots T^{N-1,0} = \mathrm{Tr}(T^N) = \sum_{i=0}^{1} \lambda_i^N \quad (2\times 2\ T)$$

λ_i - eigenvalue of T

- 1D q-state Potts model (nnb): $q \times q$ transfer matrix
- 1D Ising model with 1st & 2nd nb inter's: 4×4

So, consider an $n \times n$ T: $\lambda_0 \dots \lambda_{n-1}$ s.t. $|\lambda_0| > |\lambda_1| > \dots > |\lambda_{n-1}|$

$T|u_i\rangle = \lambda_i |u_i\rangle$

Then f (free en. per spin)

$$f = -k_B T \lim_{N\to\infty} \frac{1}{N} \log Z =$$

$$= -k_B T \lim_{N\to\infty} \frac{1}{N} \log \Big\{ \lambda_0^N \Big(1 + \sum_{i=1}^{n-1} \underbrace{\Big(\frac{\lambda_i}{\lambda_0}\Big)^N}_{\to 0} \Big\} =$$

$$= -k_B T \log \lambda_0 \quad (*)$$

λ_0 is non-degenerate

Can prove that λ_0 is non-degenerate & positive for transfer matrices

$\lambda_i (i \neq 0)$ may be complex but (*) still holds

Next, correlation f'n: [consider 1D nnb Ising model for simplicity]

$$\Gamma_R = \underset{\substack{\uparrow \\ \text{thermal average}}}{\langle S_0 S_R \rangle} - \langle S_0 \rangle \langle S_R \rangle$$

$\Gamma_R \sim e^{-R/\xi}$ when R is large:
(purely exp'l)

$$\xi^{-1} = \lim_{R\to\infty} \left\{ -\frac{1}{R} \log |\Gamma_R| \right\}$$

Consider $\langle S_0 S_R \rangle = \dfrac{\sum_{\{S\}} S_0 S_R e^{-\beta \mathcal{H}}}{\underbrace{\sum_{\{S\}} e^{-\beta \mathcal{H}}}_{Z = \sum_{i=0}^{n-1} \lambda_i^N}}$

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[~~As an easy example,~~ consider

$\langle S_0 \rangle = \cancel{\sum_{\substack{S_0 = \pm 1 \\ S_1 = \pm 1}} S_0 e^{\beta J S_0 S_1 + \beta H \frac{S_0 + S_1}{2}}} \cancel{e^{\beta J S_0 S_1 + \beta H \frac{S_0 + S_1}{2}}}$]

Introduce $\mathcal{S} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
a spin matrix

Note that $\langle s_0 | = \underline{1 \quad 0}$ are its eigenvectors
$\langle s_1 | = \underline{0 \quad 1}$
with eigenvalues $s_0 = 1$ & $s_1 = -1$ respectively

Also, recall that $\mathcal{S} = \sum_{i=0,1} |s_i\rangle s_i \langle s_i|$

Likewise, $T = \sum_{i=0,1} |u_i\rangle \lambda_i \langle u_i|$

So, $Z = \mathrm{Tr}(T^N) = \sum_{i,j=0,1} \langle s_j | u_i \rangle \lambda_i^N \langle u_i | s_j \rangle$.

Indeed, note that for any 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$,

$$\underbrace{(1\ 0)}\underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{\begin{pmatrix} a \\ c \end{pmatrix}} + \underbrace{(0\ 1)}\underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\begin{pmatrix} b \\ d \end{pmatrix}} = a+d = \mathrm{Tr}(A)$$

So, $\mathrm{Tr}(A) = \sum_{i=0,1} \langle s_i | A | s_i \rangle$

Note also that

$$Z = \sum_{j=0,1} \sum_{i_0 \dots i_{N-1}} \langle s_j | u_{i_0} \rangle \lambda_{i_0} \overbrace{\langle u_{i_0} | u_{i_1} \rangle}^{\delta_{i_0 i_1}} \lambda_{i_1} \overbrace{\langle u_{i_1} | u_{i_2} \rangle}^{\delta_{i_1 i_2}} \dots$$
$$\dots \lambda_{i_{N-1}} \langle u_{i_{N-1}} | s_j \rangle$$

It's clear that to make $\sum_{\{s\}} s_0 s_R e^{-\beta H}$ rather than Z, we need to consider

explicitly label spins

$$\sum_{k=0,1} \sum_{j=0,1} \sum_{i_0 \dots i_{N-1}} s_0 \langle s_j | u_{i_0} \rangle \lambda_{i_0} \langle u_{i_0} | u_{i_1} \rangle \dots \times$$
$$\times \langle u_{i_{R-1}} | s_k \rangle s_R \langle s_k | u_{i_R} \rangle \dots =$$

$$= \sum_{j,k} \sum_{i_0, i_1} s_0 \langle s_j | u_{i_0} \rangle \lambda_{i_0}^R \langle u_{i_0} | s_k \rangle s_R \langle s_k | u_{i_1} \rangle \times$$
$$\times \lambda_{i_1}^{N-R} \langle u_{i_1} | s_j \rangle$$

Note that

$$\sum_k |s_k\rangle\langle s_k| = 1$$

Next, we realize that

$$\sum_j \langle u_{i_1} | s_j \rangle s_0 \langle s_j | u_{i_0} \rangle =$$
$$= \langle u_{i_1} | \hat{S}_0 | u_{i_0} \rangle,$$

$$\sum_k \langle u_{i_0} | s_k \rangle s_R \langle s_k | u_{i_1} \rangle = \langle u_{i_0} | \hat{S}_R | u_{i_1} \rangle$$

$$\left\{ \begin{array}{l} i_0 \to i, \\ i_1 \to j \end{array} \right. \Rightarrow \sum_{\{s\}} s_0 s_R e^{-\beta \mathcal{H}} = \sum_{i,j} \langle u_j | \hat{S}_0 | u_i \rangle \times$$
$$\times \lambda_i^R \langle u_i | \hat{S}_R | u_j \rangle \lambda_j^{N-R}.$$

Finally,

$$\langle s_0 s_R \rangle = \frac{\sum_{i,j} \langle u_j | \hat{S}_0 | u_i \rangle \left(\frac{\lambda_i}{\lambda_0} \right)^R \langle u_i | \hat{S}_R | u_j \rangle \left(\frac{\lambda_j}{\lambda_0} \right)^N}{\sum_k \left(\frac{\lambda_k}{\lambda_0} \right)^N}$$

$N \to \infty$: only $j=0, k=0$ terms survive

$$\lim_{N \to \infty} \langle s_0 s_R \rangle = \sum_{i=0,1} \left(\frac{\lambda_i}{\lambda_0} \right)^R \langle u_0 | \hat{S}_0 | u_i \rangle \langle u_i | \hat{S}_R | u_0 \rangle =$$
$$= \overbrace{\langle u_0 | \hat{S}_0 | u_0 \rangle}^{\langle s_0 \rangle} \overbrace{\langle u_0 | \hat{S}_R | u_0 \rangle}^{\langle s_R \rangle} +$$
$$+ \left(\frac{\lambda_1}{\lambda_0} \right)^R \langle u_0 | \hat{S}_0 | u_1 \rangle \langle u_1 | \hat{S}_R | u_0 \rangle.$$

So, $\Gamma_R = \left(\frac{\lambda_1}{\lambda_0} \right)^R \langle u_0 | \hat{S}_0 | u_1 \rangle \langle u_1 | \hat{S}_R | u_0 \rangle$ depends on all eigenvalues & eigenvectors of T

$\underbrace{n>2}_{\text{dim of the transfer matrix}}$: $\Gamma_R = \sum_{i\neq 0} \left(\frac{\lambda_i}{\lambda_0}\right)^R \langle u_0 | S_0 | u_i\rangle \langle u_i | S_R | u_0\rangle$

Finally,

$$\xi^{-1} = \lim_{R\to\infty} \left[-\frac{1}{R} \log\left\{ \left(\frac{\lambda_1}{\lambda_0}\right)^R \langle u_0 | S_0 | u_1\rangle * * \langle u_1 | S_R | u_0\rangle \right\}\right] \in$$

$\uparrow$ $i=1$ term dominates

$$= -\log\left(\frac{\lambda_1}{\lambda_0}\right).$$

Involves only λ_0, λ_1

Explicit results for 1D Ising model:

$$\lambda_{0,1} = e^{\beta J}\cosh(\beta H) \pm \sqrt{e^{2\beta J}\sinh^2(\beta H) + e^{-2\beta J}}$$

$$\langle u_0| = \underbrace{\overbrace{\alpha_+ \alpha_-}}$$

$$\langle u_1| = \underbrace{\overbrace{\alpha_- \; -\alpha_+}}$$

$$\alpha_\pm^2 = \frac{1}{2}\left(1 \pm \frac{e^{\beta J}\sinh(\beta H)}{\sqrt{e^{2\beta J}\sinh^2(\beta H) + e^{-2\beta J}}}\right)$$

So, free en. per spin:

$$f = -k_B T \log(\lambda_0) = -k_B T \log\left\{ e^{\beta J}\cosh(\beta H) + \right.$$
$$\left. + \sqrt{e^{2\beta J}\sinh^2(\beta H) + e^{-2\beta J}} \right\}.$$

$\beta \to \infty$:

$$f = -k_B T \log\left\{ e^{\beta J} \frac{e^{\beta H}}{2} + e^{\beta J} \frac{e^{\beta H}}{2} \right\} =$$

$$= -k_B T \left(\beta J + \beta H \right) = \underline{\underline{-J - H}} \,.$$

↗ just energy per spin

Magnetization:

$$\langle S \rangle = \langle u_0 | S | u_0 \rangle = \underline{\alpha_+ \alpha_-} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \alpha_+ \\ \alpha_- \end{pmatrix}}_{\begin{pmatrix} \alpha_+ \\ -\alpha_- \end{pmatrix}} =$$

↑ eigenvector w/largest eigenvalue

$$= \alpha_+^2 - \alpha_-^2 = \frac{e^{\beta J} \sinh(\beta H)}{\sqrt{e^{2\beta J} \sinh^2(\beta H) + e^{-2\beta J}}} \,.$$

$$J = 0 \Rightarrow \langle S \rangle = \frac{\sinh(\beta H)}{\sqrt{1 + \sinh^2(\beta H)}} = \tanh(\beta H)$$

↗ paramagnetic phase

$H \to 0$, T finite:

$$\langle S \rangle \to 0 \,.$$

<u>But</u>: $$\lim_{H \to 0} \lim_{T \to 0} \langle S \rangle = \pm \frac{e^{\beta J} e^{\beta H}/2}{e^{\beta J} e^{\beta H}/2} = \pm 1$$

So, we have a phase transition at $T_c = 0 \Rightarrow \langle S \rangle = 0$ goes to $\langle S \rangle = \pm 1$

Corr'n f'n

$$\Gamma_R = \left(\frac{\lambda_1}{\lambda_0}\right)^R \underbrace{\alpha_+\alpha_-}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\overbrace{\begin{pmatrix}\alpha_-\\ -\alpha_+\end{pmatrix}}^{\begin{pmatrix}\alpha_-\\ \alpha_+\end{pmatrix}} *$$

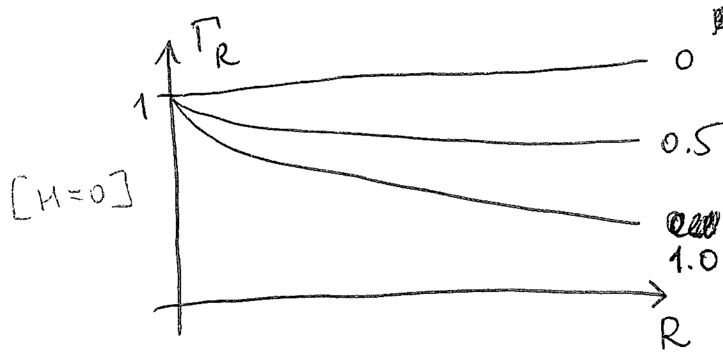
$$* \underbrace{\alpha_- \; -\alpha_+}\underbrace{\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\alpha_+\\ \alpha_-\end{pmatrix}}_{\begin{pmatrix}\alpha_+\\ -\alpha_-\end{pmatrix}} =$$

$$= \left(\frac{\lambda_1}{\lambda_0}\right)^R [2\alpha_+\alpha_-][2\alpha_-\alpha_+] =$$

$$= \left(\frac{\lambda_1}{\lambda_0}\right)^R 4\alpha_+^2\alpha_-^2 = \left(\frac{\lambda_1}{\lambda_0}\right)^R \left(1 - \frac{e^{\beta J}\sinh(\beta H)}{\sqrt{e^{2\beta J}\sinh^2(\beta H) + e^{-2\beta J}}}\right)$$

$$\times\left(1 + \frac{e^{\beta J}\sinh(\beta H)}{\sqrt{\ldots}}\right) = \left(\frac{\lambda_1}{\lambda_0}\right)^R \frac{e^{-2\beta J}}{e^{2\beta J}\sinh^2(\beta H) + e^{-2\beta J}}$$

$\underline{H=0}$: $\Gamma_R \to \left(\frac{e^{\beta J} - e^{-\beta J}}{e^{\beta J} + e^{-\beta J}}\right)^R = \tanh^R(\beta J)$.



decays ie $T>0$

Corr'n length

Finally,

$$\xi^{-1} = -\log\left(\frac{\lambda_1}{\lambda_0}\right) = -\log\left(\frac{e^{\beta J}\cosh(\beta H) - \sqrt{e^{2\beta J}\sinh^2(\beta H) + e^{-2\beta J}}}{e^{\beta J}\cosh(\beta H) + \sqrt{\ldots}}\right)$$

Consider $H=0$ for simplicity ~~$T\to\infty\ (\beta\to 0)$:~~

H, J finite
$\parallel$
0

$$\cancel{\xi^{-1} \to -\log\left(\frac{e^{\beta J}e^{\beta H/2} - e^{\beta J}e^{\beta H/2}}{e^{\beta J}e^{\beta H/2} + e^{\beta J}e^{\beta H/2}}\right) \to 0}$$

~~$\xi \to \infty \Rightarrow$ phase transition (!)~~

Indeed, $T_R(H=0) = \tanh^R(\beta J)$ means that

$$\xi^{-1} = \lim_{R\to\infty}\Big[-\frac{1}{R}\underbrace{\log T_R}_{R\log(\tanh(\beta J))}\Big] = -\log(\tanh(\beta J))$$

$\underline{\beta\to 0\ (T\to\infty)}$: $\tanh(\beta J) = \dfrac{e^{\beta J}-e^{-\beta J}}{e^{\beta J}+e^{-\beta J}} \to$

$\to \beta J$, so that

$$\xi^{-1}\to -\log(\beta J)\to +\infty$$

$\xi\to 0$ at high T $\Rightarrow$ no correl's

$\underline{\beta\to\infty\ (T\to 0)}$: $\tanh(\beta J)\to 1$, so that

$\xi^{-1}\to 0$, $\xi\to\infty$ as $T\to 0$ $\Downarrow$

phase transition (!)