HW #5

[20 points per problem]

1. Show that for a Hamiltonian

$$\mathcal{H} = -J \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j \, ,$$ where $\vec{S}_i$ is a unit vector

$\uparrow$ nnb sum (classical Heisenberg model),

the mean-field equations may be written as follows, with the trial Hamiltonian

$$\mathcal{H}_0 = -\vec{H}_0 \cdot \sum_{i=1}^{N} \vec{S}_i \, :$$

$$\begin{cases} \vec{H}_0 = Jz \langle \vec{S} \rangle_0 \, , \\ F_{mf} = -k_B T \log Z_0 + \dfrac{JzN}{2} \langle \vec{S} \rangle_0^2 \, . \end{cases}$$

$\uparrow$ mean-field free en.

Here, N = # spins, z = coordination #, $\langle \ldots \rangle_0$ = average wrt $\mathcal{H}_0$.
thermal

Finally, $$Z_0 = \int_{\Omega_1} \ldots \int_{\Omega_N} d\Omega_1 \ldots d\Omega_N \, e^{-\beta \mathcal{H}_0} =$$

$$= \left[\int_{\Omega} d\Omega \, e^{\beta \vec{H}_0 \cdot \vec{S}} \right]^N .$$

(2.) Consider the spin-1 Ising model on the 1D lattice, with $H = 0$:

$$\mathcal{H} = -J \sum_{i=1}^{N} S_i S_{i+1} , \qquad S_i = 0, \pm 1 .$$

periodic BCs

Using the transfer matrix technique, calculate f, the free energy per spin, in the thermodynamic limit. Show that it has the expected behavior in the $T \to 0$ and $T \to \infty$ limits.

(3.) Consider the spin-$\frac{1}{2}$ Ising model on the 1D lattice:

$$\mathcal{H} = -J \sum_{i=1}^{N} S_i S_{i+1} - H \sum_{i=1}^{N} S_i .$$

Using auxiliary variables

$$\begin{cases} x = e^{-4\beta J} , \\ h = \beta H \end{cases}$$

check that for small x and h, the reduced free en. per spin and the magnetization per spin can be recast in the scaling form.

Find the scaling forms and the corresponding critical exponents y_1 and y_2.