HW#1 solutions

Ex. 2.3

(1.) $X \perp Y \mid Z$ iff $p(x,y|z) = p(x|z)\,p(y|z)$.

✓ Indep. → factorization

$$p(x,y|z) = \underbrace{p(x|z)}_{\equiv\, g(x,z)}\,\underbrace{p(y|z)}_{\equiv\, h(y,z)} = g(x,z)\,h(y,z),$$ as desired

✓ Factorization → indep.

If $p(x,y|z) = g(x,z)\,h(y,z)$,

$$\sum_{x,y} p(x,y|z) = 1 = \sum_{x,y} g(x,z)h(y,z) =$$

$$= \Big(\sum_x g(x,z)\Big)\Big(\sum_y h(y,z)\Big).$$

Moreover,

$$\begin{cases} p(x|z) = \sum_y p(x,y|z) = g(x,z)\sum_y h(y,z), \\ p(y|z) = \sum_x p(x,y|z) = h(y,z)\sum_x g(x,z). \end{cases}$$

Then $p(x|z)\,p(y|z) = g(x,z)\,h(y,z)\,\times$

$$\times\,\underbrace{\Big(\sum_x g(x,z)\sum_y h(y,z)\Big)}_{=1} = p(x,y|z),$$ as desired

(2.) Ex. 2.4

$x_1 \sim \mathcal{N}(\mu_1, \sigma_1^2)$, $x_2 \sim \mathcal{N}(\mu_2, \sigma_2^2)$

$p(y) = ?$, where $y = x_1 + x_2$

Note that $\mathcal{N}(x | \mu, \sigma^2) = \varphi(x - \mu | \sigma) =$

$= \frac{1}{\sigma} \varphi\left(\frac{x-\mu}{\sigma}\right)$, where

$\varphi(z) = \text{pdf of } \mathcal{N}(0,1)$.
standard normal

Then $p(y) = \int dx_1 \frac{1}{\sigma_1} \varphi\left(\frac{x_1 - \mu_1}{\sigma_1}\right) \frac{1}{\sigma_2} \varphi\left(\frac{y - x_1 - \mu_2}{\sigma_2}\right) =$

$$= \frac{1}{2\pi \sigma_1 \sigma_2} \int dx_1 \, e^{-\frac{1}{2}\left(\frac{x_1 - \mu_1}{\sigma_1}\right)^2} e^{-\frac{1}{2}\left(\frac{y - x_1 - \mu_2}{\sigma_2}\right)^2}.$$

Complete the square:

$$\left(\frac{x_1 - \mu_1}{\sigma_1}\right)^2 + \left(\frac{y - x_1 - \mu_2}{\sigma_2}\right)^2 =$$

$$= (\omega_1 + \omega_2)(x_1 - \bar{x}) + \frac{\omega_1 \omega_2}{\omega_1 + \omega_2} (y - (\mu_1 + \mu_2))^2,$$

where $\begin{cases} \omega_1 = \sigma_1^{-2}, \\ \omega_2 = \sigma_2^{-2}. \end{cases}$ $\qquad \frac{\omega_1 \omega_2}{\omega_1 + \omega_2} = \frac{1}{\sigma_1^2 + \sigma_2^2}$

$$\bar{x} = \frac{\omega_1 \mu_1 + \omega_2 (y - \mu_2)}{\omega_1 + \omega_2}$$

Finally,

$$p(y) = \frac{1}{2\pi\sigma_1\sigma_2} e^{-\frac{1}{2(\sigma_1^2+\sigma_2^2)}(y-(\mu_1+\mu_2))^2} \times$$

$$\times \underbrace{\int dx_1\, e^{-\frac{1}{2}(\omega_1+\omega_2)(x_1-\bar{x})^2}}_{\sqrt{2\pi}\,\frac{\sigma_1\sigma_2}{\sqrt{\sigma_1^2+\sigma_2^2}}} =$$

$$= \frac{1}{\sqrt{2\pi(\sigma_1^2+\sigma_2^2)}} e^{-\frac{1}{2(\sigma_1^2+\sigma_2^2)}[y-(\mu_1+\mu_2)]^2} =$$

$$= \mathcal{N}(y\,|\,\mu_1+\mu_2, \sigma_1^2+\sigma_2^2).$$

Ex. 2.6

(3.) Variance of a sum

$$V[X+Y] = E[(X+Y)^2] - (E[X]+E[Y])^2 =$$

$$= E[X^2+Y^2+2XY] - [E[X]^2 + E[Y]^2 + 2E[X]E[Y]]$$

$$= (E[X^2] - E[X]^2) + (E[Y^2] - E[Y]^2) +$$

$$+2(E[XY] - E[X]E[Y]) =$$

$$= Var[X] + Var[Y] + 2Cov[X,Y].$$

If X & Y are indep., $Cov[X,Y]=0$

and $V[X+Y] = V[X] + V[Y]$.

4. Ex. 2.11

	X	Y	P
	G	G	1/4
(b) →	G	B	1/4
	B	G	1/4
(b) →	B	B	1/4

(a)

$$\text{(a)}\quad \underbrace{P(N_g=1 \mid N_b \geq 1)}_{\text{or read it off directly from the prob. table}} = \frac{\overbrace{P(N_g=1, N_b \geq 1)}^{= P(N_g=1,\, N_b=1)}}{P(N_b \geq 1)} = \frac{1/2}{3/4} = \frac{2}{3}.$$

$$\text{(b)}\quad \underbrace{P(X=G \mid Y=B)}_{\text{or directly from prob. table}} = \frac{P(X=G, Y=B)}{P(Y=B)} = \frac{1/4}{1/2} = \frac{1}{2}.$$