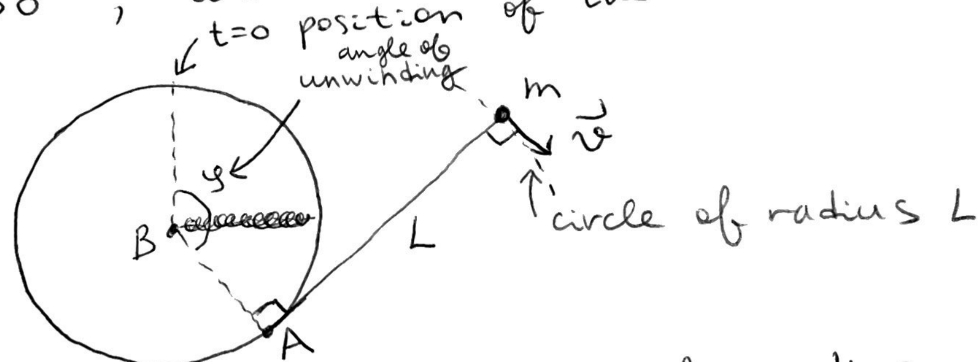
Midterm solutions (2022)

1. (a) Choose length L of the unwound cord as the generalized coordinate.

$$\mathcal{L} = T - \underset{\substack{\parallel \\ 0,\ \text{no external forces}}}{V} = T = \frac{m v^2}{2}$$

Note that $E = T + V = \dfrac{m v^2}{2}$ is conserved $\Rightarrow$ $v = v_0$, constant at any t

At $t > 0$, we have:



The mass is on a circle of radius L & the velocity $\vec{v}$ is tangential to the circle, with $|\vec{v}| = \text{const}$ as discussed above. For this system,

$v = L\omega$, where ω is the angular velocity of m on the circle of radius L. However, $\underline{\omega = \dot{\varphi}}$, where φ is the total angle of unwinding:

$$\varphi(t=0) = 0 \quad \text{and} \quad \varphi = \frac{L}{R}$$

(e.g., if $L = 2\pi R$, $\varphi = 2\pi$ etc.)

$v = L\dot{\varphi}$

Now, $$\mathcal{L} = \frac{mL^2\dot{\varphi}^2}{2} = \frac{mL^2\dot{L}^2}{2R^2}.$$

The EoM is
$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{L}}\right) = \frac{\partial \mathcal{L}}{\partial L}, \text{ or}$$

$$\frac{d}{dt}\left(\frac{\cancel{m}L^2}{\cancel{R^2}}\dot{L}\right) = \frac{\cancel{m}\dot{L}^2}{\cancel{R^2}} L,$$

$$\ddot{L}L^2 + \cancel{2}L\dot{L}^2 = \underset{0}{\cancel{L\dot{L}^2}},$$

$$\frac{d}{dt}(L\dot{L}) = 0 \quad (*)$$

(b) Define $y = \dfrac{L^2}{2} \Rightarrow \dot{y} = L\dot{L}$

$(*) \Rightarrow \ddot{y} = 0$, or $y = Bt + C$,

$$L^2 = 2Bt + 2C$$

$$L(0) = 0 \Rightarrow C = 0.$$

$$v = \underbrace{\frac{L\dot{L}}{R}}_{\dot{y}/R} \Rightarrow \dot{y} = B \text{ gives}$$

$$v_0 = \frac{B}{R}, \text{ or}$$

note that $v(0) = v_0 \neq 0$ but $L(0) = 0 \Rightarrow \dot{L}(0) = \infty$ (!)

$$L^2(t) = 2Rv_0t. \quad (**)$$

(c) Finally, the angular momentum around point A is simply

$$\ell' = m v L = \frac{m L^2 \dot{L}}{R} = \frac{m}{R} \underbrace{(2 R v_0 t)^{\frac{1}{2}}}_{L} \underbrace{R v_0}_{L\dot{L} = R v_0} =$$

$$= m (2 R v_0^3 t)^{1/2}.$$

ℓ' is not conserved around A

Note that $\ell = \ell'$ since $\vec{AB} \uparrow\uparrow \vec{v}$: $\vec{v} \times \vec{AB} = 0$.

around B, cylinder axis

Another way to get Eq. (**): note that $\vec{T} \perp \vec{v} \Rightarrow v = \text{const}$ as already discussed above

tension force in the rope

But then $d\varphi = \frac{dL}{R} = \dot{\varphi}\, dt = \frac{v_0\, dt}{L}$.

$$\dot{\varphi} = \frac{v}{L} = \frac{v_0}{L}$$

Hence $L\, dL = R v_0\, dt$, or

$L^2 = 2 R v_0 t \qquad (L(0) = 0)$

same as (**)

(2.) Recall that

$$V_{eff}(r) = \underbrace{V(r)}_{= -\frac{km}{r^n}} + \frac{\ell^2}{2mr^2}$$

Circular orbit:

$$V'_{eff}(r_0) = 0 \quad , \text{ yielding}$$

$$\frac{kmn}{r_0^{n+1}} - \frac{\ell^2}{mr_0^3} = 0 .$$

Hence $\quad km^2 n\, r_0^3 = \ell^2 r_0^{n+1}$, or

$$\underline{\underline{r_0^{n-2} = \frac{km^2 n}{\ell^2}}} .$$

Now, $\quad V''_{eff}(r_0) = -\frac{kmn(n+1)}{r_0^{n+2}} + \frac{3\ell^2}{mr_0^4} .$

$V''_{eff}(r_0) > 0$ for a stable orbit:

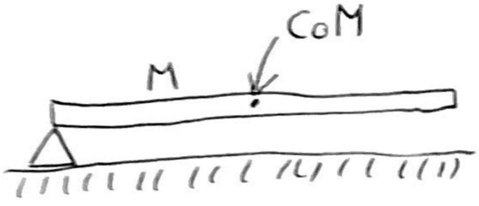
$$-\frac{kmn(n+1)}{r_0^{n-2}} + \frac{3\ell^2}{m} > 0 , \text{ or}$$

$$-\frac{\cancel{k}m\cancel{n}(n+1)}{\cancel{k}m^2\cancel{n}}\cancel{\ell^2} + \frac{3\cancel{\ell^2}}{m} > 0 ,$$

$$-(n+1) + 3 > 0 , \text{ or}$$

$$\boxed{n < 2}$$

(3.)



$t = 0^+$

Define $\ddot{x}$ = CoM linear acceleration (downward)

Clearly, $M\ddot{x} = Mg - \underbrace{F}$

force of the left support acting on the rod (equal & opposite to the force we're asked to compute)

Angular momentum:

$$\underbrace{Mg\frac{L}{2}}_{\text{torque}} = I\ddot{\theta}$$

I: moment of inertia, $= \frac{1}{3}ML^2$

θ = rod angle from the horizontal $[\theta(0)=0]$

Note that $\frac{L}{2}\theta \simeq x$ for small x, θ

$$\Downarrow$$

$$\frac{L}{2}\ddot{\theta} = \ddot{x}$$

Finally,

$$\frac{M}{I}\underbrace{\frac{\ddot{x}}{\ddot{\theta}}}_{\frac{L}{2}} = \frac{Mg-F}{Mg\frac{L}{2}} \Rightarrow \underbrace{\frac{M}{I}}_{\frac{3}{L^2}} Mg\left(\frac{L}{2}\right)^2 = Mg - F,$$

$$\frac{3}{4}Mg = Mg - F \Rightarrow \boxed{F = \frac{Mg}{4}}$$