Lecture 26 } Restricted HJ equations

If $H(q,p) = \text{const}(t)$, we have restricted HJ equation:

$$H\left(\{q_i\}, \{\underbrace{\frac{\partial W}{\partial q_i}}_{p_i}\}\right) = \underbrace{\alpha_1}_{\text{basically, } E} \qquad (*)$$

W is a CT generating function in its own right: consider a CT s.t. $P_i = \alpha_i, \forall i$.

Then if we define $W = W(q, \mathbf{P})$, we have:

$$\begin{cases} p_i = \dfrac{\partial W}{\partial q_i}, \\ Q_i = \dfrac{\partial W}{\partial P_i} = \dfrac{\partial W}{\partial \alpha_i} \end{cases} \Leftarrow \text{but this is consistent with } \begin{cases} p_i = \dfrac{\partial S}{\partial q_i} \\ Q_i = \dfrac{\partial S}{\partial \alpha_i} \end{cases}$$

from before, so that this is "our" W

However, $H(\{q_i\},\{p_i\}) = \underbrace{\alpha_1}_{\text{new canonical momentum}}$

$$\swarrow$$

$$H\left(\{q_i\}, \{\frac{\partial W}{\partial q_i}\}\right) = \alpha_1$$

W does not depend on t: $\frac{\partial W}{\partial t} = 0 \rightarrow \underbrace{K}_{\text{"new } H\text{"}} = \alpha_1$

Thus, W generates a CT in which all $P_i = \alpha_i$ are const $\Rightarrow$ all Q_i's are cyclic:

$$\begin{cases} \dot{P}_i = -\dfrac{\partial K}{\partial Q_i} = 0 \Rightarrow P_i = \alpha_i \\ \dot{Q}_i = \dfrac{\partial \overset{\sim=\alpha_1}{K}}{\partial \alpha_i} = \delta_{i1} \end{cases}$$

Then $\begin{cases} Q_1 = t + \beta_1 = \dfrac{\partial W}{\partial \alpha_1}, \\ Q_{i\neq 1} = \beta_i = \dfrac{\partial W}{\partial \alpha_i}. \end{cases}$ $(**)$ ← no t involved

All coords are const except Q_1, which is actually time (!). Thus, $H=\alpha_1$ is the generalized momentum conjugate to t.

Eq. $(*)$ is a partial differential equation with n integration constants but one is trivial: $W \to W + \alpha$; the remaining const are $\{\alpha_2, \dots, \alpha_n\} + \alpha_1$ = new const canonical momenta

Use $p_i = \dfrac{\partial W(q, \alpha)}{\partial q_i}\Big|_{t=0}$ to find $\alpha_i = \alpha_i(p_{i,0}; q_{i,0})$
$i = 1, \dots, n$

Next, we can use $(**)$ (both eq's) to get

$q_i = q_i(\alpha, \beta, t) = q_i(p_{i,0}; q_{i,0}; t) \Rightarrow$ basically $q_i(t)$, solutions of EoM

α depend on ICs

We can even choose one of q_i's at $t=0$ to

Use the second set of Eq's $(**)$ [for $i \neq 1$] to obtain $\beta_{i\neq 1} = \beta_{i\neq 1}(p_{i,0}; q_{i,0})$; use the $i=1$ eq'n to get $\beta_1 = \beta_1(p_{i,0}; q_{i,0})$

(say q_1) as an indep. variable and solve for $q_2(q_1)$, $q_3(q_1)$, ... using the lower, t-indep. Eq. (**) [for $i \neq 1$].

$\rightarrow$ orbit EoM

Separation of variables

Completely separable:

$$S = \sum_{i=1}^{n} S_i(q_i; \alpha_1, \ldots, \alpha_n; t) \quad \text{, or}$$

n HJ eq's:

$$H_j\left(q_j; \frac{\partial S_j}{\partial q_j}; \alpha_1, \ldots, \alpha_n; t\right) + \frac{\partial S_j}{\partial t} = 0$$

If H does not depend on t,

$$S_i(q_i; \alpha_1, \ldots, \alpha_n; t) =$$

$$= W_i(q_i; \alpha_1, \ldots, \alpha_n) - \alpha_i t$$

$$\Downarrow$$

n restricted HJ eq's:

$$H_i\left(q_i; \frac{\partial W_i}{\partial q_i}; \alpha_1 \ldots \alpha_n\right) = \alpha_i$$

ordinary differential eq's, 1st order

[α_i's are called separation constants]

One can show that any cyclic coordinate is separable.

Let's say q_1 is cyclic $\Rightarrow p_1 = \text{const} \equiv \gamma$

Restricted HJ equation:

$$(+) \quad H\left(q_2 \ldots q_n ; \gamma ; \frac{\partial W}{\partial q_2} \ldots \frac{\partial W}{\partial q_n}\right) = \alpha_1$$

Now, try $W = W_1(q_1, \{\alpha\}) + W'(q_2 \ldots q_n, \{\alpha\})$

Then $W \to W'$ in Eq. (+), while

$$p_1 = \gamma = \frac{\partial W_1}{\partial q_1} \Rightarrow W_1 = \gamma q_1 + \underbrace{\text{const}}_{\text{set to 0 for simplicity}}$$

So, $W = W' + \gamma q_1$

Similar to t & $-H$: if $H = \text{const}(t)$, t is cyclic and a $-Et$ term appears.

If $B < n$ coordinates are non-cyclic & fully separable:

$$W(\{q\}, \{\alpha\}) = \underbrace{\sum_{i=1}^{B} W_i(q_i, \{\alpha\})}_{\text{If fully separable}} + \underbrace{\sum_{i=B+1}^{n} q_i \alpha_i}_{\text{cyclic coords}}$$

$$\Downarrow$$

$$H_i\left(q_i ; \frac{\partial W_i}{\partial q_i} ; \underbrace{\alpha_1, \alpha_2, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_n}_{\{\alpha\}\ (?)}\right) = \alpha_i \qquad i = 1, \ldots, B$$

easy to solve, as before

Action-angle variables : 1 DoF

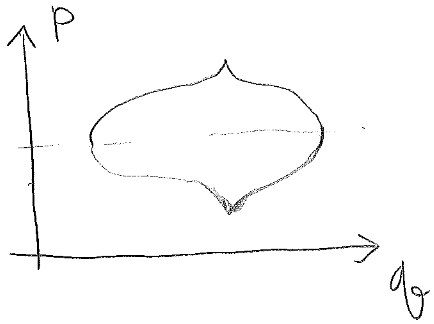
Consider a conservative system:

$$H(q, p) = \alpha_1$$
$$\Downarrow$$
$$p = p(q_\theta, \alpha_1)$$

orbit equation in 2D phase space

Focus on 2 types of periodic motion:

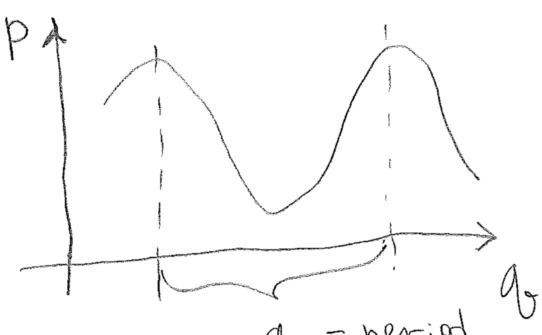
1. Closed orbits ("libration")



e.g. 1D HO

2. Rotation

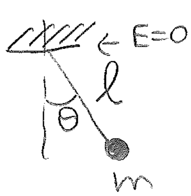
e.g. a rigid body rotating around an axis



(some angle)

$q_{\theta 0}$ = period (2π)

Ex.



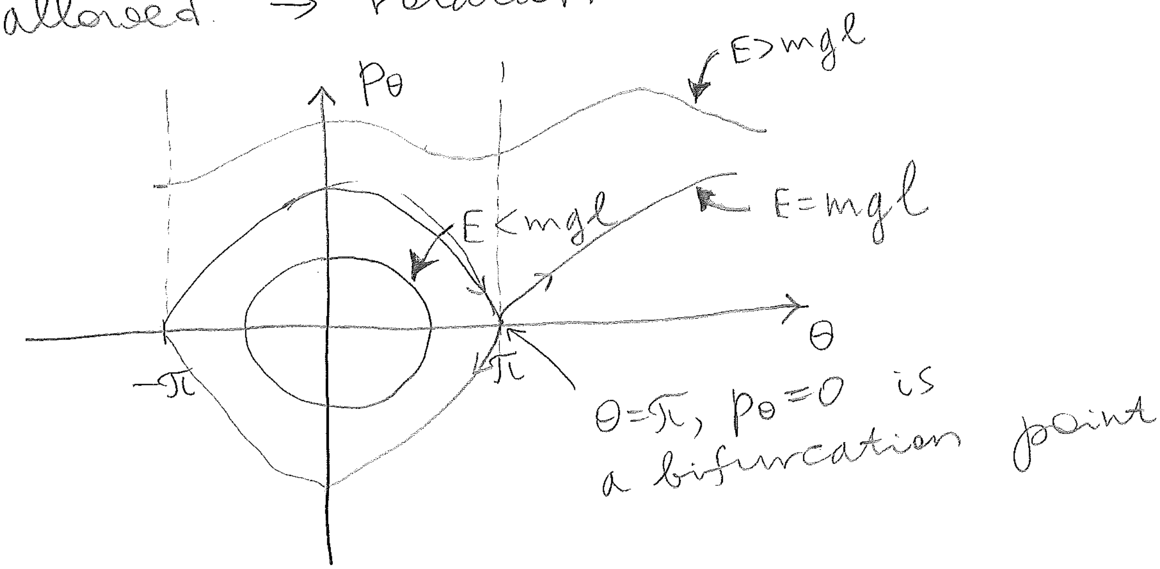
$$E = \frac{p_\theta^2}{2m\ell^2} - mg\ell\cos\theta \text{ , or}$$

$$p_\theta = \pm\sqrt{2m\ell^2(E + mg\ell\cos\theta)}$$

If $E < mgl$, the motion is confined between $-\theta_m$ & θ_m, where $\cos\theta_m = -\frac{E}{mgl}$

libration $\qquad\qquad$ $p_{\theta_m} = 0$

If $E > mgl$, all values of θ are allowed $\rightarrow$ rotation



Now, introduce action variable:

$$J = \oint p\, dq_\theta$$

over a period, definite $\int$

$$J = \oint dq_\theta\, p(q_\theta, \alpha_1) = \underbrace{F}_{\text{some f'n}}(\alpha_1).$$

$[J]$ = ang. momentum

Invert: $\alpha_1 = F^{-1}(J)$

But $H = \alpha_1 \Rightarrow F^{-1}(J) = H(J) = \alpha_1$

Then $W = W(q_\theta, J)$ ← instead of α_1

Then $\frac{\partial W}{\partial J} = \underset{\text{angle var (conjugate to } J)}{w}$ (1)

$\dot{w} = \frac{\partial H(J)}{\partial J} = \nu(J) \Rightarrow w = \nu t + \beta.$

Could solve Eq. (1): $q = q(\underset{\nu t + \beta}{\underset{\shortparallel}{w}}, J)$

$\downarrow$

$q = q(t)$

Instead, consider

$$\underbrace{\Delta w}_{\substack{\text{change over} \\ \text{a period}}} \equiv \oint dq \frac{\partial w}{\partial q} = \oint dq \frac{\partial^2 W}{\partial q \partial J} =$$

$$= \frac{d}{dJ} \oint dq \underbrace{\frac{\partial W}{\partial q}}_{p} = \frac{d}{dJ} \underbrace{\oint dq\, p}_{J} = \underline{\underline{1}}.$$

~~Δw~~ Then $\underset{1}{\underset{\shortparallel}{\Delta w}} = \nu \underbrace{\Delta t}_{\shortparallel\, \tau \text{ here}}$

$$\nu = \frac{1}{\tau}$$

ν = freq. of the periodic motion of q

Consider 1D HO:

$$J = \oint p\, dq = \oint dq \sqrt{2mE - m^2 \tilde{\omega}^2 q^2}$$

$$E = \frac{p^2}{2m} + \frac{m \tilde{\omega}^2 q^2}{2}$$

Substitute $q = \sqrt{\frac{2E}{m\tilde{\omega}^2}} \sin\theta$:

$$J = \oint dq \sqrt{2mE - \frac{m^{\cancel{2}} \tilde{\omega}^2}{\cancel{m} \tilde{\omega}^2} 2E \sin^2\theta} =$$

one cycle in θ

$$= \oint dq \sqrt{2mE} \cos\theta = \frac{2E}{\tilde{\omega}} \underbrace{\int_0^{2\pi} d\theta \cos^2\theta}_{\pi} = \frac{2\pi E}{\tilde{\omega}}$$

$$dq = \sqrt{\frac{2E}{m\tilde{\omega}^2}} \cos\theta\, d\theta$$

$$\langle \cos^2\theta \rangle = \frac{\int_0^{2\pi} d\theta \cos^2\theta}{2\pi} = \frac{1}{2}$$

Then $E \equiv H = \frac{J\tilde{\omega}}{2\pi}$

$$\nu(J) = \frac{\partial H}{\partial J} = \underbrace{\frac{\tilde{\omega}}{2\pi}}_{} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \Rightarrow \tilde{\omega} = 2\pi\nu$$

linear frequency of a linear HO [obtained w/out solving EoM]

Finally, recall that

Canon. transform from (J, ω) to (p, q)

$$\underset{\nu t + \beta}{\omega} \quad q(t) = \sqrt{\frac{2E}{m\tilde{\omega}^2}} \sin(\tilde{\omega} t + \beta) = \sqrt{\frac{\cancel{2} J \tilde{\omega}}{\cancel{2}\pi \tilde{\omega}^2 m}} \sin(2\pi\nu_t + \underset{2\pi\beta}{\beta'}) =$$

$$= \sqrt{\frac{J}{\pi m \tilde{\omega}}} \sin(2\pi\nu_t + \beta') \quad " 2\pi\beta$$

Likewise, $p(t) = \sqrt{\frac{mJ\tilde{\omega}}{\pi}} \cos(2\pi\nu t + \beta')$

$\quad " m\dot{q}(t)$