Liouville's theorem

Lecture **25**.

Suppose that the system is governed by classical mechanics but the number of particles is too great to keep track of all the microscopic trajectories. Then, even if we knew all the initial conditions, we cannot obtain the solution for later times. The best we can do is to impose some constraints (such as the total $E = \text{const}$) and consider an ensemble of systems satisfying these constraints. Each system is a point in phase space at any given t => the ensemble of systems is a distribution of such points.

Now, consider $D(q, p)$, density of such systems in the neighborhood of (q, p).

At equilibrium, $D(q, p)$ does not depend on t.

We can formally consider the motion of the systems as that of incompressible "liquid" in phase space, and apply the continuity eq'n to it:

(the total # systems stays const with t)

3D: $\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v}) = 0 \quad \Rightarrow \quad \vec{\nabla}\cdot(\rho\vec{v}) = 0$ at steady state

In phase space, this generalizes to

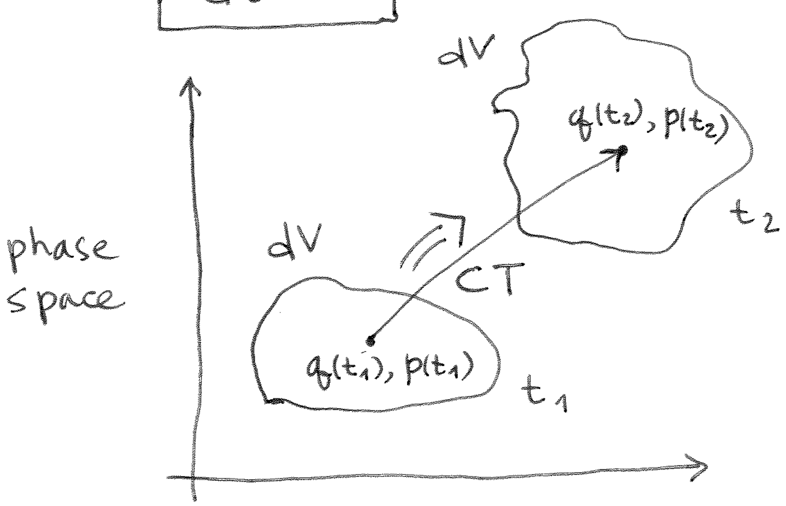
$$\underbrace{\frac{\partial}{\partial q_i}(D\dot{q}_i) + \frac{\partial}{\partial p_i}(D\dot{p}_i)}_{\text{implicit sums}} = 0 \text{ , or}$$

$$\dot{q}_i \frac{\partial D}{\partial q_i} + \dot{p}_i \frac{\partial D}{\partial p_i} + D\Big[\underbrace{\frac{\partial \dot{q}_i}{\partial q_i}}_{\frac{\partial^2 H}{\partial q_i \partial p_i}} + \underbrace{\frac{\partial \dot{p}_i}{\partial p_i}}_{-\frac{\partial^2 H}{\partial p_i \partial q_i}}\Big] = 0 .$$

Then $\dot{q}_i \frac{\partial D}{\partial q_i} + \dot{p}_i \frac{\partial D}{\partial p_i} = \frac{dD}{dt} = 0$

$\uparrow$ $\frac{\partial D}{\partial t} = 0$ at equil.

$\boxed{\frac{dD}{dt} = 0}$ Liouville's theorem



Shown before that infinitesimal volume around (q,p) does not change in size (but in general changes its shape)

But the number of systems inside the volume also does not change: if a system were to cross the border from the inside it would be infinitesimally close to some system on the border. But then they will have to "travel together" from that point on. So the system cannot cross the border from within (i.e. leave the volume) or from the outside (i.e. come into the volume). But that means that both

$dV = \text{const}$ & $\underbrace{dN(q,p)}_{\text{\# systems within the volume}}$ is const: $D = \dfrac{dN}{dV}$ is const as well, as shown above

Hamilton - Jacobi theory

Idea: as before, simplify Hamiltonian H through a canonical transformation to new variables that are const in time:

$$\begin{cases} q(t), \\ p(t) \end{cases} \Rightarrow \begin{cases} q_0, \\ \underbrace{p_0}_{\text{ICs at } t=0} \end{cases}$$

This transformation, if found, can be used to get $\begin{cases} q = q(q_0, p_0, t) \\ p = p(q_0, p_0, t) \end{cases}$ $\Leftarrow$ the solution of the mechanical problem

Recall that $K = H + \dfrac{\partial F}{\partial t}$, and require

$K = 0$ ("as simple as possible").

Then $\begin{cases} \dfrac{\partial K}{\partial P_i} = \dot{Q}_i = 0, \\ \dfrac{\partial K}{\partial Q_i} = -\dot{P}_i = 0 \end{cases} \Rightarrow \begin{cases} Q_i = \text{const}, \\ P_i = \text{const}, \end{cases} \forall i$ as desired

So, $H(q, p, t) + \dfrac{\partial F}{\partial t} = 0$.

Take $F = F_2(q, P, t)$, then $\overbrace{p_i}^{\text{old momenta}} = \dfrac{\partial F_2}{\partial q_i}$

and $H\left(q_1, \ldots, q_n; \dfrac{\partial F_2}{\partial q_1} \ldots \dfrac{\partial F_2}{\partial q_n}; t\right) + \dfrac{\partial F_2}{\partial t} = 0.$

$\Uparrow$ Hamilton-Jacobi (HJ) eq'n

[partial differential eq'n of $\underbrace{(q_1, \ldots, q_n; t)}_{n+1 \text{ vars}}$]

F_2 is typically called S, Hamilton's principal function. Suppose we have

$$S = S(q_1, \ldots, q_n; \underbrace{\alpha_1, \ldots, \alpha_{n+1}}_{n+1}; t)$$

$n+1$ const of integration, expected b/c HJ eq'n only involves partial derivatives of S and b/c physically we want to relate α_i's & P_i's, the new const momenta

Note that if S is a solution, $S+\alpha$ is also a solution $\Rightarrow$ say $\alpha = \alpha_{n+1}$.

Discarding α as uninteresting, we're left with

$$S = S(q_1, \ldots, q_n; \underbrace{\alpha_1, \ldots, \alpha_n}_{\text{can take } P_i = \alpha_i,\ \forall i}; t)$$

Then $p_i = \dfrac{\partial S(q, \alpha, t)}{\partial q_i} \Rightarrow$ can obtain $\underbrace{\alpha_i}_{\text{new momenta}} = \alpha_i(p, q, t)$

Next, $\underbrace{Q_i}_{\parallel\ \beta_i = \text{const}} = \dfrac{\partial S(q, \alpha, t)}{\partial \alpha_i} \Rightarrow$ can obtain

$\beta_i = \beta_i(p, q, t)$

or

$\underbrace{\beta_i = \beta_i(\alpha, q, t)}_{\text{can invert this to obtain}}$

$$q_j = q_j(\alpha, \beta, t)$$

Finally, we have $p_j = p_j(\alpha, \beta, t)$

complete solution of EoM

Note that α_i's & β_i's are not automatically $p_{0,i}$'s & $q_{0,i}$'s, but they can be related to IC's.

Now, examine

$$\frac{dS}{dt} = \underbrace{\frac{\partial S}{\partial q_i}}_{p_i} \dot{q}_i + \underbrace{\frac{\partial S}{\partial t}}_{-H} = p_i \dot{q}_i - H = \mathcal{L} \quad , \text{ or}$$

$\uparrow$ $P_i = \text{const}, \ \forall i$

$$S = \underbrace{\int dt\, \mathcal{L}}_{\text{indefinite } \int} + \text{const}$$

If $\dfrac{\partial H}{\partial t} = 0$, $\quad \underbrace{H = E}_{\text{conserved total energy}}$

Then $\quad S = \int dt \left(p_i \dot{q}_i - H \right) = \underbrace{\int dt\, p_i \dot{q}_i}_{=\int dq_i\, p_i \,\equiv\, W(q,\alpha)} - Et$

$\{ p_i = p_i(t),\ q_i = q_i(t) \}$ can be viewed as $p_i = p_i(q)$ and then $\int dq_i\, p_i(q)$ is just a f'n of q_i's (and α_i's)

note that explicit t-dependence is gone from W since

$\underbrace{W(q,\alpha)}$ $\Rightarrow$ Hamilton's characteristic f'n

So, $\quad S(q, \alpha, t) = W(q, \alpha) - Et \quad$, where

$$\frac{dW}{dt} = \underbrace{\frac{\partial W}{\partial q_i}}_{p_i} \dot{q}_i = p_i \dot{q}_i \quad , \text{ consistent with above}$$

Harmonic oscillator

(1D)

Consider $$H = \frac{1}{2m}(p^2 + m^2\omega^2 q^2) = E$$

$\boxed{n=1}$ $$\omega = \sqrt{\frac{k}{m}}$$

HJ eq'n

$$p = \frac{\partial S}{\partial q} \Rightarrow \frac{1}{2m}\left[\left(\frac{\partial S}{\partial q}\right)^2 + m^2\omega^2 q^2\right] + \frac{\partial S}{\partial t} = \underbrace{0}_{\text{new Hamiltonian, } K}$$

$n=1$: expect one $\underbrace{\alpha}_{\text{new momentum}}(=E)$ & one $\underbrace{\beta}_{\text{new coord}}$

Use $S = W - Et$:

const of integration

$$\frac{1}{2m}\left[\left(\frac{\partial W}{\partial q}\right)^2 + m^2\omega^2 q^2\right] = E \text{ , or}$$

$$S = \sqrt{2mE}\int dq \sqrt{1 - \frac{m\omega^2 q^2}{2E}} - Et$$

$$\Uparrow$$

$$\frac{\partial W}{\partial q} = \sqrt{2mE - m^2\omega^2 q^2} =$$

$$= \sqrt{2mE}\sqrt{1 - \frac{m\omega^2 q^2}{2E}}.$$

Now, $$\underbrace{\beta}_{\text{new coord}} = \frac{\partial S}{\partial E} = \int dq \frac{1}{2\sqrt{2mE - m^2\omega^2 q^2}} 2m - t =$$

$$= \sqrt{\frac{m}{2E}}\int dq \frac{2m}{\sqrt{\frac{2m}{E}}\sqrt{2mE - m^2\omega^2 q^2}} - t =$$

$$= \sqrt{\frac{m}{2E}}\int dq \frac{1}{\sqrt{1 - \frac{m\omega^2 q^2}{2E}}} - t \text{, or}$$

$$t + \beta = \frac{1}{\omega}\sin^{-1}\left[q\sqrt{\frac{m\omega^2}{2E}}\right]. \quad (*)$$

Invert (*):

$$q = \sqrt{\frac{2E}{m\omega^2}}\ \sin(\omega t + \underbrace{\omega\beta}_{\beta' = \text{const}})$$

$\beta' =$ const , osc. phase

Next, $\underbrace{p}_{\text{old momentum}} = \frac{\partial S}{\partial q} = \frac{\partial W}{\partial q} = \sqrt{2mE - m^2\omega^2 q^2}$.

Substitute q:

$$p = \sqrt{2mE - m^2\omega^2 \frac{2E}{m\omega^2}\sin^2(\omega t + \beta')} =$$

$$= \sqrt{2mE}\ \cos(\omega t + \beta') = m\dot{q}\ ,$$ as expected.

How are E & β' related to $p(0) = p_0$ & $q(0) = q_0$?

$$p_0^2 + m^2\omega^2 q_0^2 = 2mE\ ,$$ energy conservation

Finally, at $t = 0$

$$\tan\beta' = m\omega\frac{q_0}{p_0}\ .$$

Thus, $\begin{cases} q_0 = 0 \\ p_0 \neq 0 \end{cases} \Rightarrow \beta' = 0$ osc. starts at its equil. position at $t=0$

So, the new coords are (i) total E, (ii) initial phase of the oscillator

Lastly,

$$S = 2E \int dt \cos^2(\omega t + \beta') - Et =$$

$$= \int dt \underbrace{2E \left(\cos^2(\omega t + \beta') - \frac{1}{2} \right)}_{\mathcal{L}}$$

Indeed, $\mathcal{L} = \frac{1}{2m} (p^2 - m^2 \omega^2 q^2) =$

$$= E \left(\cos^2(\omega t + \beta') - \sin^2(\omega t + \beta') \right) =$$

$$= 2E \left(\cos^2(\omega t + \beta') - \frac{1}{2} \right),$$ just as above