Symplectic approach to canonical transformations

Consider $\begin{cases} Q_i = Q_i(p,q) \\ P_i = P_i(p,q) \end{cases}$ (*) restricted canonical transform

Recall that $H(Q,P,t)$ is obtained from $H(q,p,t)$ by substituting (i.e. it does not change under restricted transforms)

$$\begin{cases} q_j = q_j(P,Q) \\ p_j = p_j(P,Q) \end{cases} \quad (**) \Leftarrow \text{inverse of } (*)$$

Then $$\dot{Q}_i = \frac{\partial Q_i}{\partial q_j}\dot{q}_j + \frac{\partial Q_i}{\partial p_j}\dot{p}_j = \frac{\partial Q_i}{\partial q_j}\frac{\partial H}{\partial p_j} - \frac{\partial Q_i}{\partial p_j}\frac{\partial H}{\partial q_j}.$$

On the other hand,

$$\frac{\partial H}{\partial P_i} = \frac{\partial H}{\partial p_j}\frac{\partial p_j}{\partial P_i} + \frac{\partial H}{\partial q_j}\frac{\partial q_j}{\partial P_i}.$$

Thus, $\dot{Q}_i = \frac{\partial H}{\partial P_i}$ iff $$\begin{cases} \left(\frac{\partial Q_i}{\partial q_j}\right)_{q,p} = \left(\frac{\partial p_j}{\partial P_i}\right)_{Q,P}, \\ \left(\frac{\partial Q_i}{\partial p_j}\right)_{q,p} = -\left(\frac{\partial q_j}{\partial P_i}\right)_{Q,P} \end{cases}$$

Similarly, use $\dot{P}_i = -\frac{\partial H}{\partial Q_i}$ to obtain:

$$\begin{cases} \left(\frac{\partial P_i}{\partial q_j}\right)_{q,p} = -\left(\frac{\partial p_j}{\partial Q_i}\right)_{Q,P}, \\ \left(\frac{\partial P_i}{\partial p_j}\right)_{q,p} = \left(\frac{\partial q_j}{\partial Q_i}\right)_{Q,P}. \end{cases}$$

Now use symplectic notation:

$$\dot{\vec{\eta}} = J \frac{\partial H}{\partial \vec{\eta}} \quad \Leftarrow H = H(\vec{\eta}, t)$$

Consider a restricted canonical transform $\vec{\xi} = \vec{\xi}(\vec{\eta})$.

Now, $\dot{\xi}_i = \underbrace{\frac{\partial \xi_i}{\partial \eta_j}}_{M_{ij}} \dot{\eta}_j \quad \Rightarrow \quad \dot{\vec{\xi}} = \underbrace{M}_{\text{Jacobian matrix}} \dot{\vec{\eta}}$.

Next, $\dot{\vec{\xi}} = M J \frac{\partial H}{\partial \vec{\eta}}$

If we change vars in H, $H = H(\vec{\xi}, t)$,

we obtain: $\frac{\partial H}{\partial \eta_i} = \frac{\partial H}{\partial \xi_j} \underbrace{\frac{\partial \xi_j}{\partial \eta_i}}_{M_{ji} = (\tilde{M})_{ij}}$, or

$$\frac{\partial H}{\partial \vec{\eta}} = \tilde{M} \frac{\partial H}{\partial \vec{\xi}}.$$

Finally, $\dot{\vec{\xi}} = \underbrace{M J \tilde{M}}_{J \text{ since the transform'n is canonical}} \frac{\partial H}{\partial \vec{\xi}}$

So, $\underline{M J \tilde{M} = J}^{(1)} \Rightarrow M J = J \tilde{M}^{-1}$, or

$\underbrace{J(M J)\underbrace{(-J)}_{J^{-1}}}_{\mathbb{I}} = \underbrace{J (J \tilde{M}^{-1})(-J)}_{\underbrace{(-J)J}_{\mathbb{I}} (\tilde{M}^{-1} J)} \Rightarrow \underline{\underline{J M = \tilde{M}^{-1} J}}.$

Finally, $\underline{\tilde{M} J M = J}$. $\quad (2)$

(1) & (2) are symplectic conditions for a canonical transform'n.

Ex. Consider $n=2$ $\quad \vec{\eta} = \begin{pmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{pmatrix}$ and $\vec{\xi} = \begin{pmatrix} Q_1 \\ Q_2 \\ P_1 \\ P_2 \end{pmatrix}$

and the generating f'n

$$F' = q_1 P_1 + q_2 Q_2$$

leads to $\begin{cases} Q_1 = q_1 , & P_1 = p_1 \\ Q_2 = p_2 , & P_2 = -q_2 \end{cases}$

Then
$$\begin{pmatrix} \dot{Q}_1 \\ \dot{Q}_2 \\ \dot{P}_1 \\ \dot{P}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}}_{M} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{p}_1 \\ \dot{p}_2 \end{pmatrix}$$

On the other hand, $\dot{\vec{\xi}} = J \dfrac{\partial H}{\partial \vec{\xi}}$ yields

$$\begin{pmatrix} \dot{Q}_1 \\ \dot{Q}_2 \\ \dot{P}_1 \\ \dot{P}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}}_{J} \begin{pmatrix} \frac{\partial H}{\partial Q_1} \\ \frac{\partial H}{\partial Q_2} \\ \frac{\partial H}{\partial P_1} \\ \frac{\partial H}{\partial P_2} \end{pmatrix} .$$

It's easy to check that (1) & (2) are satisfied.

Now consider $\vec{\xi} = \vec{\xi}(\vec{\eta}, t)$ and focus first on an infinitesimal canonical transform: $\vec{\xi} = \vec{\eta} + \delta\vec{\eta} \Rightarrow \begin{cases} Q_i = q_i + \delta q_i, \\ P_i = p_i + \delta p_i \end{cases}$

The generating function is

$$\underbrace{F_2(q, P, t) = q_i P_i}_{\text{identity transform}} + \underset{\substack{\uparrow \\ \text{small prm}}}{\epsilon}\, G(q, P, t).$$

Then
$$\begin{cases} p_j = \dfrac{\partial F_2}{\partial q_j} = P_j + \epsilon \dfrac{\partial G}{\partial q_j} \Rightarrow \delta p_j = -\epsilon \dfrac{\partial G}{\partial q_j} \\ Q_j = \dfrac{\partial F_2}{\partial P_j} = q_j + \epsilon \underbrace{\dfrac{\partial G}{\partial P_j}} \Rightarrow \delta q_j = \epsilon \dfrac{\partial G}{\partial p_j} \end{cases}$$

to $\mathcal{O}(\epsilon)$, simply replace $\{P\}$ by $\{p\}$ everywhere in G & replace $\dfrac{\partial G}{\partial P_j} \to \dfrac{\partial G}{\partial p_j}$

$\Downarrow$ $G = G(q, p, t)$

Finally, $\delta\vec{\eta} = \epsilon J \dfrac{\partial G}{\partial \vec{\eta}}$ and

$$M = \frac{\partial \vec{\xi}}{\partial \vec{\eta}} = \mathbb{I} + \frac{\partial \delta\vec{\eta}}{\partial \vec{\eta}} \underset{\substack{\Uparrow \\ \delta\vec{\eta} = \epsilon J \frac{\partial G}{\partial \vec{\eta}}}}{=} \mathbb{I} + \epsilon J \underbrace{\frac{\partial^2 G}{\partial \vec{\eta}\, \partial \vec{\eta}}}.$$

symm. matrix, ij element: $\left(\dfrac{\partial^2 G}{\partial \eta_i \partial \eta_j} \right)$

Next, $\tilde{M} = \mathbb{1} + \epsilon \widetilde{\left(\frac{\partial^2 G}{\partial \vec{\eta} \partial \vec{\eta}}\right)} \underbrace{\tilde{J}}_{-J} = \mathbb{1} - \epsilon \frac{\partial^2 G}{\partial \vec{\eta} \partial \vec{\eta}} J$.

But then

$$M J \tilde{M} = \left(\mathbb{1} + \epsilon J \frac{\partial^2 G}{\partial \vec{\eta} \partial \vec{\eta}}\right) J \left(\mathbb{1} - \epsilon \frac{\partial^2 G}{\partial \vec{\eta} \partial \vec{\eta}} J\right) \simeq$$

$$\underset{\Theta(\epsilon)}{\simeq} J + \epsilon J \frac{\partial^2 G}{\partial \vec{\eta} \partial \vec{\eta}} J - \epsilon J \frac{\partial^2 G}{\partial \vec{\eta} \partial \vec{\eta}} J = J.$$

Finally, the reasoning above applies to

$$\vec{\xi}(t_0) \to \vec{\xi}(t_0 + \underbrace{dt}_{\text{acts as } \epsilon})$$

Then $\vec{\xi}(t_0) \to \vec{\xi}(t)$ also satisfies the symplectic condition if we build it up in steps of dt.

But the transformation $\vec{\eta} \to \vec{\xi}(t_0)$ is symplectic since it is time-independent.

So, if both $\vec{\eta} \to \vec{\xi}(t_0)$ & $\vec{\xi}(t_0) \to \vec{\xi}(t)$ are canonical, so is $\vec{\eta} \to \vec{\xi}(t)$. Thus any canonical transform, time-dependent or not, satisfies the symplectic conditions (1) & (2).

It can be shown that canonical transformations form a group, with group multiplication defined as 2 successive canonical transfotmations.

Poisson brackets and Canonical invariants

Poisson bracket is defined as

$$[u,v]_{q,p} = \frac{\partial u}{\partial q_i}\frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial v}{\partial q_i} .$$

In matrix form,

$$[u,v]_{\vec{\eta}} = \widetilde{\frac{\partial u}{\partial \vec{\eta}}}\, J\, \frac{\partial v}{\partial \vec{\eta}}$$

Note that

$$\begin{cases} [q_j, q_k]_{q,p} = \dfrac{\partial q_j}{\partial q_i}\underbrace{\dfrac{\partial q_k}{\partial p_i}}_{=0} - \underbrace{\dfrac{\partial q_j}{\partial p_i}}_{=0}\dfrac{\partial q_k}{\partial q_i} = 0, \\ [p_j, q_k]_{q,p} = \underbrace{\dfrac{\partial p_j}{\partial q_i}\dfrac{\partial q_k}{\partial p_i}}_{=0} - \underbrace{\dfrac{\partial p_j}{\partial p_i}}_{\delta_{ji}}\underbrace{\dfrac{\partial q_k}{\partial q_i}}_{\delta_{ki}} = -\delta_{jk}, \\ [p_j, p_k]_{q,p} = 0, \\ [q_j, p_k]_{q,p} = \delta_{jk}. \end{cases}$$

In matrix form,

$$\underbrace{[\vec{\eta}, \vec{\eta}]_{\vec{\eta}}}_{[\eta_l, \eta_m]_{\vec{\eta}} \text{ is the } l,m \text{ element of this matrix}} = J .$$

Now consider $\vec{\xi} = \vec{\xi}(\vec{\eta}, t)$, a time-dependent canonical transformation $(p, q) \to (P, Q)$.

In matrix language,

$$[\vec{\xi}, \vec{\xi}]_{\vec{\eta}} = \widetilde{\frac{\partial \vec{\xi}}{\partial \vec{\eta}}} J \underbrace{\frac{\partial \vec{\xi}}{\partial \vec{\eta}}}_{M,\ \text{Jacobian matrix}} = \widetilde{M} J M = J. \quad (*)$$

Conversely, if $(*)$ is valid the $\vec{\eta} \to \vec{\xi}$ transform must be canonical.

Since $[\vec{\xi}, \vec{\xi}]_{\vec{\xi}} = J$, $(*)$ implies that Poisson brackets of canonical vars themselves (called <u>fundamental</u> PBs) are <u>invariant</u> under canonical transformations. This is equivalent to $\widetilde{M} J M = J$, the symplectic condition of a canonical transform.

Now consider
$$\begin{cases} \dfrac{\partial v}{\partial \vec{\eta}} = \widetilde{M} \dfrac{\partial v}{\partial \vec{\xi}}, \\[2ex] \widetilde{\dfrac{\partial u}{\partial \vec{\eta}}} = \widetilde{\widetilde{M} \dfrac{\partial u}{\partial \vec{\xi}}} = \widetilde{\dfrac{\partial u}{\partial \vec{\xi}}} M \end{cases}$$

Hence
$$[u, v]_{\vec{\eta}} = \widetilde{\frac{\partial u}{\partial \vec{\eta}}} J \frac{\partial v}{\partial \vec{\eta}} = \widetilde{\frac{\partial u}{\partial \vec{\xi}}} \underbrace{M J \widetilde{M}}_{J} \frac{\partial v}{\partial \vec{\xi}} =$$

$$= [u, v]_{\vec{\xi}}.$$

So, <u>all</u> Poisson brackets are canonical invariants.

To emphasize that, we shall drop the subscripts : $[u,v]_{\vec{z}} \to [u,v]$ and so on.

Note that $[u,u] = [v,v] = 0$,

$[u,v] = -[v,u]$ (antisymmetry).

Furthermore,

$$[au+bv, w] = a[u,w] + b[v,w]$$
(linearity)

$$[uv, w] = [u,w]v + u[v,w]$$
(distributive property)

Finally,

$$[u,[v,w]] + [v,[w,u]] + [w,[u,v]] = 0$$
Jacobi's identity

Other canonical invariants: Lagrange bracket $\{u,v\}$, defined as

$$\{u,v\}_{q,p} = \frac{\partial q_i}{\partial u}\frac{\partial p_i}{\partial v} - \frac{\partial p_i}{\partial u}\frac{\partial q_i}{\partial v}, \text{ or}$$

$$\{u,v\}_{\vec{\eta}} = \widetilde{\frac{\partial \vec{\eta}}{\partial u}} J \frac{\partial \vec{\eta}}{\partial v}$$
in matrix notation

Fundamental Lagrange brackets:

$$\begin{cases} \{q_i, q_j\}_{qp} = 0, & \{p_i, p_j\}_{qp} = 0, \\ \{q_i, p_j\}_{qp} = \delta_{ij} \end{cases}$$

In matrix notation, $\{\vec{\eta}, \vec{\eta}\} = J$

One can use the Jacobian and the symplectic condition to show that $\{u, v\}$ is canonically inv.

Lagrange & Poisson brackets are "inverses" of one another, in the following sense:

Consider u_i $(i = 1, \ldots, 2n)$, $2n$ indep. f's of canonical vars q_k & p_k $(k = 1, \ldots, n)$.

Then $\{\vec{u}, \vec{u}\}$ is a $2n \times 2n$ matrix with $\{u_i, u_j\}$ as the i,j th element. Similarly, $[\vec{u}, \vec{u}]$ is a $2n \times 2n$ matrix. Then it can be shown that

$$\{\vec{u}, \vec{u}\}[\vec{u}, \vec{u}] = -\mathbb{1}_{2n}.$$

Lagrange brackets do <u>not</u> obey Jacobi's identity.

Magnitude of a volume element in phase space is canonically inv:

Consider $\vec{\eta} \rightarrow \vec{\xi}$ [2n-dim phase space]

Then volume element

$$(d\eta) = dq_1\, dq_2 \ldots dq_n\, dp_1 \ldots dp_n$$

$$\Downarrow$$

$$(d\xi) = dQ_1 \ldots dQ_n\, dP_1 \ldots dP_n$$

Now, $(d\xi) = \underbrace{\|M\|}_{\text{absolute value of det of Jacobian matrix}} (d\eta)$

But the symplectic condition yields:

$$|M|^2 |J| = |J| \Rightarrow |M| = \pm 1, \text{ or}$$
$$\|M\| = 1.$$

Then $(d\xi) = (d\eta)$ and therefore

$V_n = \int \dots \int (d\eta)$ is a canonical invariant

volume of an arbitrary region of phase space

If $n=1$, $(d\eta) = dq\,dp$ & $V_1 = \int dq\,dp$