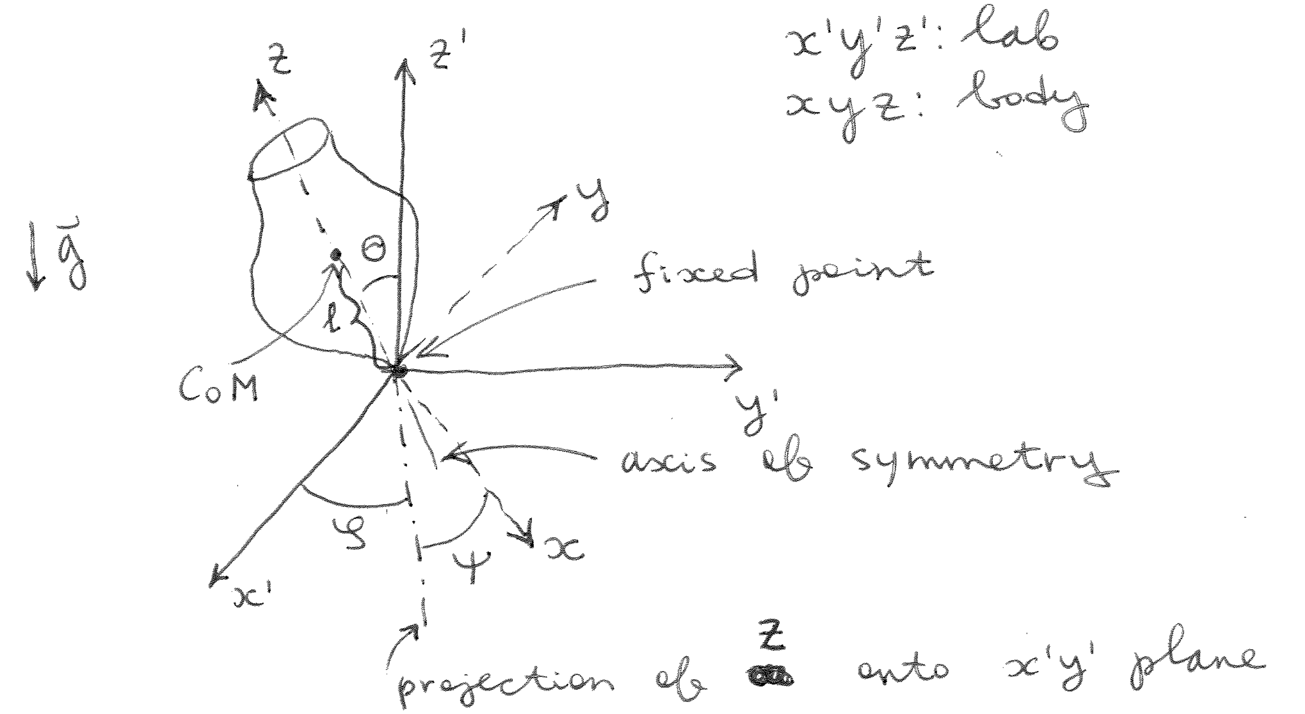
Lecture 14

Heavy symmetrical top with one point fixed



Euler angles: (φ, ψ, θ)

Note that z is a principal axis. Also, $I_1 = I_2 \neq I_3$.

$\dot{\theta}$ = nutation, "bobbing" up and down of z (and the top) wrt z',

$\dot{\varphi}$ = precession, rotation of z around z' (imagine that θ is fixed),

$\dot{\psi}$ = rotation of the top around z (body principal axis)

Often, $\dot{\psi} \gg \dot{\theta}, \dot{\varphi}$ in practice.

Use the Lagrangian approach:

$$T = \frac{I_1}{2}(\omega_1^2 + \omega_2^2) + \frac{I_3}{2}\omega_3^2$$

see 4.9

It can be shown that

in the body frame $\Rightarrow$
$$\begin{cases} \omega_1 = \dot{\varphi}\sin\theta\sin\psi + \dot{\theta}\cos\psi, \\ \omega_2 = \dot{\varphi}\sin\theta\cos\psi - \dot{\theta}\sin\psi, \\ \omega_3 = \dot{\varphi}\cos\theta + \dot{\psi}. \end{cases}$$

Then $\omega_1^2 + \omega_2^2 = \underline{\dot{\varphi}^2\sin^2\theta\sin^2\psi} + \dot{\theta}^2\cos^2\psi +$

$+ 2\dot{\varphi}\dot{\theta}\sin\theta\cancel{\sin\psi\cos\psi} + \underline{\dot{\varphi}^2\sin^2\theta\cos^2\psi} +$

$+ \dot{\theta}^2\sin^2\psi - 2\dot{\varphi}\dot{\theta}\cancel{\sin\theta\cos\psi\sin\psi} =$

$= \dot{\varphi}^2\sin^2\theta + \dot{\theta}^2.$

Therefore,

$$T = \frac{I_1}{2}(\dot{\varphi}^2\sin^2\theta + \dot{\theta}^2) + \frac{I_3}{2}(\dot{\varphi}\cos\theta + \dot{\psi})^2$$

Furthermore, the potential energy of any body is

$$V = -\sum_i m_i \vec{r}_i\cdot\vec{g} = -M\vec{R}\cdot\vec{g}.$$

In this case, $V = Mgl\cos\theta$, and

$$\mathcal{L} = \frac{I_1}{2}\left(\dot\theta + \dot\varphi^2 \sin^2\theta\right) + \frac{I_3}{2}\left(\dot\psi + \dot\varphi\cos\theta\right)^2 - Mgl\cos\theta .$$

Note that φ and ψ are cyclic coords:

$$p_\psi = \frac{\partial \mathcal{L}}{\partial \dot\psi} = I_3 \underbrace{(\dot\psi + \dot\varphi\cos\theta)}_{\omega_3} \overset{(1)}{=} I_3\,\omega_3 \equiv I_1 a = \text{const},$$

yielding $\omega_3 = \text{const}$

Next,

$$p_\varphi = \frac{\partial \mathcal{L}}{\partial \dot\varphi} = \left(I_1 \sin^2\theta + I_3\cos^2\theta\right)\dot\varphi +$$

$$+ \underset{(2)}{I_3 \dot\psi \cos\theta} \equiv I_1 b = \text{const}$$

Finally,

$$E = T + V = \frac{I_1}{2}\left(\dot\theta^2 + \dot\varphi^2\sin^2\theta\right) + \underbrace{\frac{I_3}{2}\omega_3^2}_{\text{const}} + Mgl\cos\theta = \text{const}.$$

Now,

$$I_3\dot\psi = I_1 a \overset{(1)}{} - I_3\dot\varphi\cos\theta \quad \text{and}$$

$$I_1\dot\varphi\sin^2\theta + \underset{(2)}{I_1 a\cos\theta} = I_1 b, \quad \text{yielding}$$

$$\dot\varphi = \frac{b - a\cos\theta}{\sin^2\theta}.$$

Thus, if we knew $\theta(t)$, we could find $\dot\varphi$ & then $\dot\psi$:

$$\dot{\psi} = \frac{I_1}{I_3} a - \frac{b - a\cos\theta}{\sin^2\theta}\cos\theta .$$

We can get an eq'n for θ alone:

$$\underbrace{E - \frac{I_3}{2}\omega_3^2}_{E' = \text{const}} = \frac{I_1 \dot{\theta}^2}{2} + \underbrace{\frac{I_1}{2}\frac{(b - a\cos\theta)^2}{\sin^2\theta} + Mgl\cos\theta}_{V'(\theta),\ \text{effective 1D potential}} \qquad (*)$$

effective 1D problem for θ

Define 4 normalized constants:

$$\begin{cases} \alpha = \dfrac{2E - I_3\omega_3^2}{I_1}, \\ \beta = \dfrac{2Mgl}{I_1}, \\ a = \dfrac{p_\psi}{I_1}, \\ b = \dfrac{p_\varphi}{I_1} \end{cases}$$

Then $(*)$ becomes

$$\alpha = \dot{\theta}^2 + \frac{(b - a\cos\theta)^2}{\sin^2\theta} + \beta\cos\theta .$$

Next, introduce $u = \cos\theta$:

$$\dot{u} = -\sin\theta \times \dot{\theta}, \qquad \dot{u}^2 = \sin^2\theta \times \dot{\theta}^2$$

$$\alpha \underbrace{\sin^2\theta}_{1-u^2} = \underbrace{\dot\theta^2\sin^2\theta}_{\dot u^2} + (b - a\underbrace{\cos\theta}_{u})^2 + \beta \underbrace{\sin^2\theta}_{1-u^2} \times \underbrace{\cos\theta}_{u}, \text{ or}$$

$$\dot u^2 = (\alpha - \beta u)(1-u^2) - (b - a u)^2 =$$

$$= \beta u^3 - (\alpha + a^2)u^2 + (2ab - \beta)u + (\alpha - b^2) \equiv f(u)$$

Note that $\beta \geq 0$, with $\beta = 0$ iff $\ell = 0 \Rightarrow$ the fixed point is the CoM, as in gyroscopes. Then $f(u)$ is quadratic. Here we will focus on $\underline{\underline{\beta > 0}}$.

$f(u) = 0$ (i.e. the roots of the cubic eq'n) provide information about the turning points.

Physically, $-1 \leq \underset{\cos\theta}{\underset{\shortparallel}{u}} \leq 1$. Moreover, if the top is on a horizontal surface, $\underline{u > 0}$ since $\cos\theta$ is never negative. Then turning points of u are the same as turning points of θ since $\sin\theta$ does not change sign between 0 & $\frac{\pi}{2}$

Let's formally consider $f(u)$ in the $(-\infty, +\infty)$ range. Clearly, for $|u| \gg 1$

$f(u) \simeq \beta u^3$ [pos. for $u>0$ & neg. for $u<0$]

Next, focus on $u = \pm 1$:

$$f(1) = \beta - (\alpha + a^2) + (2ab - \beta) + (\alpha - b^2) =$$
$$= -(a-b)^2 < 0 .$$

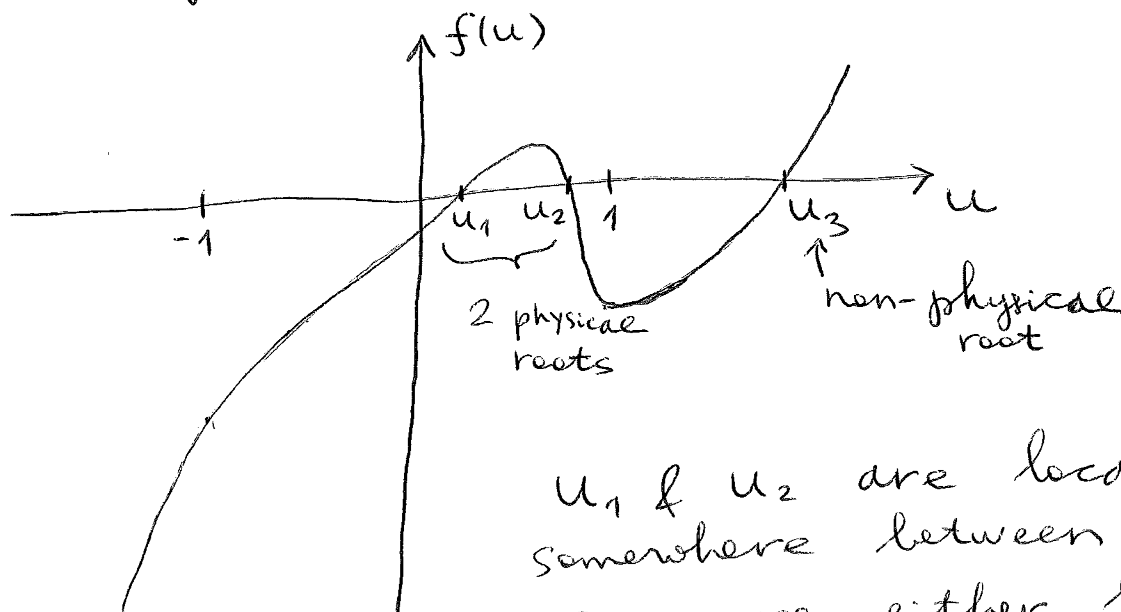
Likewise,

$$f(-1) = -\beta - (\alpha + a^2) - (2ab - \beta) + (\alpha - b^2) =$$
$$= -(a+b)^2 < 0 .$$

The only exception is $\theta = 0$ (vertical top) because then

$$\begin{cases} p_\psi = I_3(\dot\psi + \dot\varphi) = I_1 a, \\ p_\varphi = I_3 \dot\varphi + I_3 \dot\psi = I_1 b \end{cases} \Rightarrow a = b$$

This means that $f(1) = 0$ and $u = 1$ is a root. If $f(1)$ is not a root, $f(u)$ qualitatively looks like this:



u_1 & u_2 are located somewhere between -1 & $+1$; $\cos\theta$ moves either between u_1 & u_2 or between u_{min} & u_2, where u_{min} is given by the max angle θ_{max}

allowed by the horizontal surface constraint.

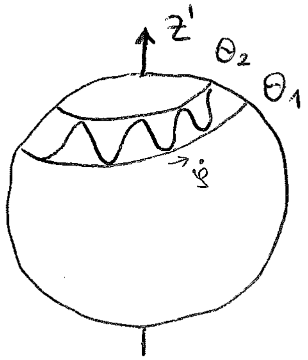
Top's motion is often visualized as a trace its z-axis would have left on a unit sphere centered on the fixed point. This trace is known as the <u>locus</u> of the z-axis. The locus lies between 2 circles: $\theta_1 = \arccos u_1$ & $\theta_2 = \arccos u_2$ $[u_1 < u_2]$ $[\theta_2 < \theta_1]$, at which $\dot\theta = 0$. The shape of the locus curve is largely determined by $u' = \frac{b}{a}$.

Suppose that $u' > u_2$, then

$$\dot\varphi = \frac{b - a\overbrace{\cos\theta}^{u \le u_2}}{\sin^2\theta} = \frac{u' - u}{\underbrace{1-u^2}_{>0 \text{ between } \theta_1 \,\&\, \theta_2}}\, a > 0$$

↑ precession velocity

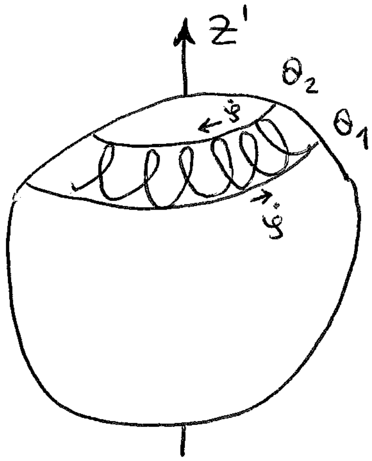
$\dot\varphi > 0$ means that φ only increases with time:



$\dot\varphi > 0$ everywhere

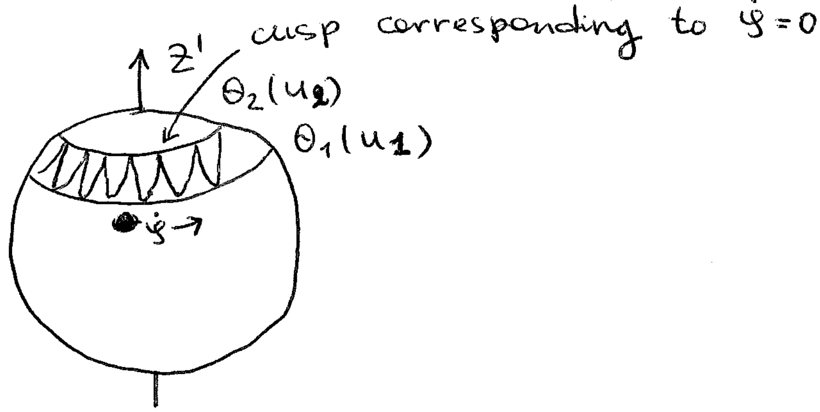
In general, the top executes nutation (changes in θ) & precession (changes in φ) as it rotates around its own z-axis.

If $u_1 < u' < u_2$, $\dot{\varphi} > 0$ at u_1 but $\dot{\varphi} < 0$ at u_2. In general, $\dot{\varphi}$ does not vanish on average, so there's still precession overall:



What if $u' = u_2$? (for example; $u' = u_1$ is similar)

Then $\dot{\varphi} = 0$ if $u = u_2$ and $\dot{\varphi} > 0$ if $u = u_1$:



Consider a top spinning around its z-axis (inclined at some angle θ_0 wrt z') at $t = 0$: $\theta = \theta_0$, $\dot{\theta} = 0$, $\dot{\varphi} = 0$ are the initial conditions.

Since $\dot{\theta} = 0$, $\dot{u} = 0$ and $u_0 = \cos\theta_0$ is automatically a root of $f(u)$.

Since $\dot{\varphi} = 0$, $b - a u_0 = 0$, or

$$u_0 = \frac{b}{a} = u'.$$

Now, $E' = E - \frac{I_3}{2}\omega_3^2 = Mgl\cos\theta_0$ at $t=0$. As $t \uparrow$, $\dot{\theta}$ & $\dot{\varphi}$ become non-zero, and the corresponding kinetic energy becomes positive. This can only be accomplished if $V \downarrow$ (i.e., $\theta \uparrow$) since the total energy is conserved. Thus $\theta_0 = \theta_2$ which corresponds to u_2, meaning that

$$u_0 = u' = u_2.$$

In other words, the top always starts to fall after it's released and continues to fall until $\theta = \theta_1$, at which point it rebounds back to θ_2. Precession is always in one direction; the locus is shown in the last Fig. above.