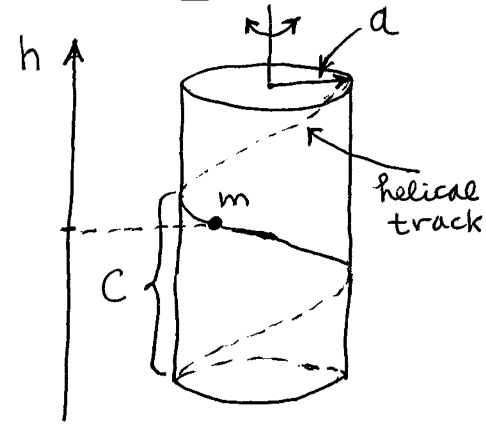
HW#7 solutions

8.24



mass density ρ

Define θ, the rotational angle of the particle wrt cylinder. That is, if the cylinder were not rotating, the point particle would slide from top to bottom and the angle would go through a certain range depending on the number of coils in the helix.

Moreover, the height of the particle is completely determined by θ:

$h\downarrow$ as $\theta\nearrow$ $\rightarrow$ $h = -c\,\theta$, where c is the vertical distance between any point on the helix and the closest point on the helix directly below (i.e., the distance between a given point and the closest point on the next helical coil, see Fig.).

Next, define φ as the rotational angle of the cylinder itself.

Then $T_{cyl} = \frac{1}{2} I \dot{\varphi}^2$, where

$I = \frac{Ma^2}{2}$ is the cylinder's moment of inertia $(M = \rho \times \pi a^2 \bar{h})$

cylinder's height

Furthermore,

$$T_{part} = \overbrace{\frac{ma^2}{2} (\dot{\theta} + \dot{\varphi})^2}^{\text{horizontal}} + \overbrace{\frac{mc^2}{2} \dot{\theta}^2}^{\text{vertical}}$$

Finally, $V_{part} = mgh = -mgc\theta$.

All together,

$$\mathcal{L} = \underbrace{\frac{I\dot{\varphi}^2}{2} + \frac{ma^2}{2}(\dot{\theta}+\dot{\varphi})^2 + \frac{mc^2}{2}\dot{\theta}^2}_{\mathcal{L}_2} + \underbrace{mgc\theta}_{\mathcal{L}_0}.$$

In matrix form,

$\mathcal{L}_2 = \frac{1}{2} \dot{\vec{q}}\, T\, \dot{\vec{q}}$, where $\vec{q} = \begin{pmatrix} \theta \\ \varphi \end{pmatrix}$ and

$$T = \begin{pmatrix} m(a^2+c^2) & ma^2 \\ ma^2 & I + ma^2 \end{pmatrix}.$$

We can use (8.27) now:

$$H = \frac{1}{2} \underbrace{\vec{p}}_{\begin{pmatrix} p_\theta \\ p_\varphi \end{pmatrix}} T^{-1} \vec{p} - \mathcal{L}_0.$$

Here, $T^{-1} = \dfrac{\tilde{T}_c}{|T|} = \dfrac{1}{m(a^2+c^2)(I+ma^2) - m^2a^4}$ $\otimes$

T_c is the cofactor matrix, see (8.28)

$$\otimes \begin{pmatrix} I+ma^2 & -ma^2 \\ -ma^2 & m(a^2+c^2) \end{pmatrix}.$$

More explicitly,

$$H = \frac{1}{2|T|}\left[(I+ma^2)p_\theta^2 + m(a^2+c^2)p_\varphi^2 - 2ma^2 p_\theta p_\varphi \right] - mgc\theta.$$

[note that φ is cyclic]

EoM: $$\begin{cases} \dot{p}_\theta = -\dfrac{\partial H}{\partial \theta} = mgc, \\ \dot{p}_\varphi = -\dfrac{\partial H}{\partial \varphi} = 0. \end{cases}$$

$$\begin{cases} \dot{\theta} = \dfrac{\partial H}{\partial p_\theta} = \dfrac{1}{|T|}\left[(I+ma^2)p_\theta - ma^2 p_\varphi\right], \\ \dot{\varphi} = \dfrac{\partial H}{\partial p_\varphi} = \dfrac{1}{|T|}\left[m(a^2+c^2)p_\varphi - ma^2 p_\theta\right]. \end{cases}$$

BCs: $p_\theta(0) = 0$, $p_\varphi(0) = 0$

$$\Downarrow$$

$$\begin{cases} p_\theta(t) = mgct, \\ p_\varphi(t) = 0. \end{cases}$$

Next, $$\begin{cases} \dot{\theta} = \dfrac{(I+ma^2)}{|T|} mgct, \\ \dot{\varphi} = -\dfrac{ma^2}{|T|} mgct. \end{cases}$$

Finally, with $\theta(0)=0$ & $\varphi(0)=0$ we obtain:

$$\begin{cases} \theta = \dfrac{(I+ma^2)\, mgc\, t^2}{2\left[m(a^2+c^2)(I+ma^2) - m^2a^4\right]}, \\[2ex] \varphi = -\dfrac{m^2a^2gct^2}{2\left[m(a^2+c^2)(I+ma^2) - m^2a^4\right]}. \end{cases}$$

Since $I = \dfrac{Ma^2}{2}$, $\quad I+ma^2 = \dfrac{a^2}{2}(2m+M)$

and $|T| = mc^2\dfrac{a^2}{2}(2m+M) + ma^2\dfrac{Ma^2}{2} =$

$= \dfrac{ma^2}{2}\left[(2m+M)c^2 + Ma^2\right]$, so that

$$\begin{cases} \theta = \dfrac{(2m+M)gct^2}{(2m+M)c^2+Ma^2}, \\[2ex] \varphi = -\dfrac{mgct^2}{(2m+M)c^2+Ma^2}. \end{cases}$$

9.2 $\underline{\underline{n=1}}$

Find M from $\begin{pmatrix} Q \\ P \end{pmatrix} = \underbrace{M}_{\text{Jacobian matrix}} \begin{pmatrix} q \\ p \end{pmatrix}$:

$$M = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}.$$

Here, $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Consequently,

$$\tilde{M} J M = \tilde{M} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} =$$

$$= \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \sin\alpha & \cos\alpha \\ -\cos\alpha & \sin\alpha \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \underline{\underline{J}}.$$

Symplectic condition is satisfied.

For the generating function, let's attempt to find $F_1(q, Q)$ (no explicit time dependence)

Let's express $p = p(q, Q)$:

$Q = q\cos\alpha - p\sin\alpha$ gives

$$p = q\cot\alpha - \frac{Q}{\sin\alpha}$$

But $p = \frac{\partial F_1}{\partial q} \Rightarrow F_1 = \frac{q^2\cot\alpha}{2} - \frac{qQ}{\sin\alpha} + f_1(Q)$. (*)

Next, $P = q\sin\alpha + p\cos\alpha =$

$$= \underbrace{q\sin\alpha + \frac{q\cos^2\alpha}{\sin\alpha}}_{q/\sin\alpha} - Q\cot\alpha \;[= P(q, Q)]$$

But $P = -\frac{\partial F_1}{\partial Q} \Rightarrow F_1 = -qQ\sin\alpha + \frac{Q^2}{2}\cot\alpha -$

$$- qQ\frac{\cos^2\alpha}{\sin\alpha} + f_2(q) =$$

$$= \frac{Q^2}{2}\cot\alpha - \frac{qQ}{\sin\alpha} + f_2(q). \quad (**)$$

Together, (*) & (**) yield

$$F_1(q, Q) = -\frac{qQ}{\sin\alpha} + \frac{q^2 + Q^2}{2}\cot\alpha.$$

This diverges if $\alpha = n\pi$, $n \in \mathbb{Z}$.
So, $\alpha \neq n\pi$ & we cannot use it for $\alpha = 0$ as requested.

Try $F_2(q, P)$ instead:

$$p = \frac{P}{\cos\alpha} - q\tan\alpha$$

$$p = \frac{\partial F_2}{\partial q} \Rightarrow F_2 = \frac{Pq}{\cos\alpha} - \frac{q^2}{2}\tan\alpha + f_1(P) \quad (1)$$

Next,

$$Q = q\cos\alpha - \left[\frac{P}{\cos\alpha} - q\tan\alpha\right]\sin\alpha =$$

$$= q\underbrace{\left[\cos\alpha + \frac{\sin^2\alpha}{\cos\alpha}\right]}_{\frac{1}{\cos\alpha}} - P\tan\alpha .$$

Then $Q = \frac{\partial F_2}{\partial P}$

$$F_2 = \frac{qP}{\cos\alpha} - \frac{P^2}{2}\tan\alpha + f_2(q) \quad (2)$$

Together, (1) & (2) imply that

$$\underline{\underline{F_2 = \frac{qP}{\cos\alpha} - \frac{q^2+P^2}{2}\tan\alpha}} .$$

This diverges for $\alpha = (n+\frac{1}{2})\pi$, $n \in \mathbb{Z}$.

So, $\alpha \neq (n+\frac{1}{2})\pi$ and we need to use F_2 to discuss $\alpha = 0$ but F_1 to discuss $\alpha = \frac{\pi}{2}$.

Physically, $\alpha = 0 \Rightarrow Q = q,\ P = p$ (identity transform)

$\alpha = \frac{\pi}{2} \Rightarrow Q = -p,\ P = q$ (exchange transform)

9.6 (a) Find $M = \begin{pmatrix} \dfrac{\partial Q}{\partial q} & \dfrac{\partial Q}{\partial p} \\ \dfrac{\partial P}{\partial q} & \dfrac{\partial P}{\partial p} \end{pmatrix}$. $\qquad$ $\underline{\underline{n=1}}$

Here,

$$\begin{cases} \dfrac{\partial Q}{\partial q} = \dfrac{\cos p}{2q^{1/2}(1+q^{1/2}\cos p)}, \\[2ex] \dfrac{\partial Q}{\partial p} = \dfrac{-q^{1/2}\sin p}{1+q^{1/2}\cos p}, \\[2ex] \dfrac{\partial P}{\partial q} = \sin(2p) + \dfrac{\sin p}{q^{1/2}}, \\[2ex] \dfrac{\partial P}{\partial p} = 2q^{1/2}\cos p + q(2\cos(2p)). \end{cases}$$

One can show by direct substitution that $\tilde{M} J M = J$, where $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Indeed,

$$JM = \begin{pmatrix} \sin(2p) + \dfrac{\sin p}{q^{1/2}} & 2q^{1/2}\cos p + 2q\cos(2p) \\[2ex] -\dfrac{\cos p}{2q^{1/2}(1+q^{1/2}\cos p)} & \dfrac{q^{1/2}\sin p}{1+q^{1/2}\cos p} \end{pmatrix},$$

and

$$\tilde{M} J M = \begin{pmatrix} \dfrac{\cos p}{2q^{1/2}(1+q^{1/2}\cos p)} & \sin(2p) + \dfrac{\sin p}{q^{1/2}} \\[2ex] -\dfrac{q^{1/2}\sin p}{1+q^{1/2}\cos p} & 2q^{1/2}\cos p + 2q\cos(2p) \end{pmatrix} \times$$

$$\times \begin{pmatrix} JM \\ \text{from above} \end{pmatrix}.$$

Clearly, $(\tilde{M} J M)_{11} = (\tilde{M} J M)_{22} = 0$, while

$$\underbrace{(\tilde{M} J M)_{12} = 1 \;, \quad (\tilde{M} J M)_{21} = -1}_{\text{some tedious algebra... (but straightforward)}} .$$

So, the symplectic condition is satisfied $\Rightarrow$

$\Rightarrow$ (Q, P) are canonical vars if (q, p) are.

(b) Consider $F_3(p, Q) = -(e^Q - 1)^2 \tan p$.

Then
$$\begin{cases} q = -\dfrac{\partial F_3}{\partial p} = \dfrac{(e^Q-1)^2}{\cos^2 p}, \\ P = -\dfrac{\partial F_3}{\partial Q} = 2(e^Q-1)e^Q \tan p . \end{cases}$$

Solve for $Q(q, p)$:

$$e^Q - 1 = q^{1/2} \cos p \quad \Rightarrow \quad Q = \log(1 + q^{1/2} \cos p).$$

Then $P = 2 \tan p \,(1 + q^{1/2} \cos p)\, q^{1/2} \cos p =$

$$= 2(1 + q^{1/2} \cos p)\, q^{1/2} \sin p .$$

Thus both P & Q are obtained using F_3.

(4.) Define $\begin{cases} u_i = \dfrac{\partial u}{\partial \eta_i}, \\ u_{ij} = \dfrac{\partial^2 u}{\partial \eta_i \, \partial \eta_j} \end{cases}$ and the same for v & w

Then $[v, w] = v_i \underbrace{J_{ij}}_{\text{element of } J} w_j$,

$$[u,[v,w]] = u_i J_{ij} [v,w]_j = u_i J_{ij} (v_k J_{kl} w_l)_j =$$

$$= u_i J_{ij} (v_k J_{kl} w_{lj} + v_{kj} J_{kl} w_l).$$

Likewise,

$$[v,[w,u]] = v_i J_{ij} (w_k J_{kl} u_{lj} + w_{kj} J_{kl} u_l),$$

$u \to v$, $v \to w$, $w \to u$

$$[w,[u,v]] = w_i J_{ij} (u_k J_{kl} v_{lj} + u_{kj} J_{kl} v_l).$$

$v \to w$, $w \to u$, $u \to v$

Now, consider

$$u_i J_{ij} v_k J_{kl} w_{lj} + \underbrace{v_i J_{ij} w_{kj} J_{kl} u_l}_{v_k J_{kl} w_{jl} J_{ji} u_i \text{, just by renaming dummy indices}} =$$

$$= u_i (\overbrace{J_{ij} + J_{ji}}^{=0}) v_k J_{kl} w_{lj} = \underline{\underline{0}}.$$

$w_{jl} = w_{lj}$ in 2^{nd} term

Likewise,

$$u_i J_{ij} v_{kj} J_{kl} w_l + \underbrace{w_i J_{ij} u_k J_{kl} v_{lj}}_{w_l J_{lk} u_i J_{ij} \underbrace{v_{jk}}_{=v_{kj}}} =$$

$$= u_i J_{ij} v_{kj} \underbrace{(J_{kl} + J_{lk})}_{=0} w_l = \underline{\underline{0}}.$$

Finally,

$$v_i J_{ij} w_k J_{kl} u_{lj} + \underbrace{w_i J_{ij} u_{kj} J_{kl} v_l}_{w_k J_{kl} \underbrace{u_{jl}}_{u_{lj}} J_{ji} v_i} =$$

$$= v_i \underbrace{(J_{ij} + J_{ji})}_{=0} w_k J_{kl} u_{lj} = \underline{\underline{0}}.$$

Thus,

$$[u,[v,w]] + [v,[w,u]] + [w,[u,v]] = 0$$

as desired.