HW #6 solutions

(1.) We shall use the normal modes to solve the problem.

As in class, we introduce η_1, η_2, η_3 as generalized coordinates:

$$\begin{cases} V = \frac{k}{2}(\eta_2-\eta_1)^2 + \frac{k}{2}(\eta_3-\eta_2)^2, \\ T = \frac{m}{2}(\dot\eta_1^2 + \dot\eta_2^2 + \dot\eta_3^2) \end{cases}$$

Then $\mathcal{L} = T - V = \frac{1}{2} T_{ij}\dot\eta_i \dot\eta_j - \frac{1}{2}V_{ij}\eta_i\eta_j$, with

$$T = \begin{pmatrix} m & & 0 \\ & m & \\ 0 & & m \end{pmatrix} \qquad V = \begin{pmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{pmatrix}$$

EoM in matrix form:

$$T\ddot{\vec\eta} + V\vec\eta = 0, \quad \text{where } \vec\eta = \begin{pmatrix}\eta_1\\ \eta_2 \\ \eta_3\end{pmatrix}$$

and the T & V matrices are defined above.

Secular eq'n:

$$|V - \omega^2 T| = 0, \text{ yielding}$$

$$\begin{cases} \omega_1 = 0, \\ \omega_2 = \sqrt{\frac{k}{m}}, \\ \omega_3 = \sqrt{\frac{3k}{m}} \end{cases} \Leftarrow \text{note that this is the } m = M \text{ result from the class}$$

The corresponding eigenvectors (again from class, and properly normalized via $\tilde{A} T A = \mathbb{I}$) are:

$$\underbrace{\vec{a}_1 = \frac{1}{\sqrt{3m}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}_{\text{rigid displacement mode}} \qquad \vec{a}_2 = \frac{1}{\sqrt{2m}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$\vec{a}_3 = \frac{1}{\sqrt{6m}} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

Finally, $\vec{\eta} = A \vec{\xi}$, where

$\vec{\xi} = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix}$ are normal coords.

Now apply the force:

$$\ddot{\xi}_j + \omega_j^2 \xi_j = f_j(t) \quad (*)$$, where

$$f_j(t) = \tilde{\vec{a}}_j \cdot \vec{F}(t) \quad \text{and} \quad \vec{F}(t) = \begin{pmatrix} f \cos \omega t \\ 0 \\ 0 \end{pmatrix}$$

So,
$$\begin{cases} f_1(t) = \dfrac{1}{\sqrt{3m}} f \cos \omega t, \\ f_2(t) = \dfrac{1}{\sqrt{2m}} f \cos \omega t, \\ f_3(t) = \dfrac{1}{\sqrt{6m}} f \cos \omega t. \end{cases}$$

Now solve eq's (*) with initial conditions $\begin{cases} \xi_i = 0 \\ \dot{\xi}_i = 0 \end{cases}$ $i = 1, 2, 3$

Try $\xi_i^p = B_i \cos(\omega t)$, $\Leftarrow$ particular sol'n where

$$
\begin{cases}
B_1 = \dfrac{\frac{1}{\sqrt{3m}} f}{-\omega^2}, \\[2ex]
B_2 = \dfrac{\frac{1}{\sqrt{2m}} f}{\omega_2^2 - \omega^2}, \\[2ex]
B_3 = \dfrac{\frac{1}{\sqrt{6m}} f}{\omega_3^2 - \omega^2}.
\end{cases}
$$

The solution to the homogen. eq's is

$$\xi_i^h = \underbrace{C_i}_{\text{real}} \cos(\omega_i t).$$

Combining the two, we get:

$$
\begin{cases}
\xi_1 = C_1 - \dfrac{f}{\sqrt{3m}\,\omega^2} \cos \omega t, \\[2ex]
\xi_2 = C_2 \cos \omega_2 t + \dfrac{f}{\sqrt{2m}\,(\omega_2^2 - \omega^2)} \cos \omega t, \\[2ex]
\xi_3 = C_3 \cos \omega_3 t + \dfrac{f}{\sqrt{6m}\,(\omega_3^2 - \omega^2)} \cos \omega t.
\end{cases}
$$

Imposing the initial conditions, we obtain:

$$\begin{cases} \xi_1 = \dfrac{f}{\sqrt{3m}\,\omega^2}(1-\cos\omega t), \\[2ex] \xi_2 = \dfrac{f}{\sqrt{2m}\,(\omega_2^2-\omega^2)}(\cos\omega t - \cos\omega_2 t), \\[2ex] \xi_3 = \dfrac{f}{\sqrt{6m}\,(\omega_3^2-\omega^2)}(\cos\omega t - \cos\omega_3 t). \end{cases}$$

Note that $\dot{\xi}_1 = \dot{\xi}_2 = \dot{\xi}_3 = 0$ is satisfied automatically at $t=0$ b/c the initial phase of the force is 0, which enabled us to use $\begin{cases} \xi_i^p = B_i \cos\omega t \\ \xi_i^h = C_i \cos(\omega_i t) \end{cases}$ (i.e., w/out the initial phases)

Finally, from $\vec{\eta} = A\vec{\xi}$ we obtain:

$$\eta_3(t) = \frac{f}{m}\left[\frac{1}{3\omega^2}(1-\cos\omega t) - \frac{1}{2(\omega_2^2-\omega^2)} \times \right.$$

$$\left. \times (\cos\omega t - \cos\omega_2 t) + \frac{1}{6(\omega_3^2-\omega^2)}(\cos\omega t - \cos\omega_3 t)\right].$$

(2.) For $0 \leq t < \alpha$,

$$\ddot{x} = \frac{F_0}{m}\frac{t}{\alpha}\cos(\omega t), \quad \text{so that}$$

$$\dot{x}(t) = \int_0^t dt' \, \frac{F_0}{m}\frac{t'}{\alpha}\cos(\omega t') =$$

$$= \frac{F_0}{m\alpha}\left[\frac{\cos(\omega t')}{\omega^2} + \frac{t'\sin(\omega t')}{\omega}\right]\Bigg|_0^t =$$

$$= \frac{F_0}{m\alpha}\left[\frac{\cos(\omega t)-1}{\omega^2} + \frac{t\sin(\omega t)}{\omega}\right].$$

Next,

$$x(t) = \int_0^t dt' \, \dot{x}(t') = \frac{F_0}{m\alpha}\left[\frac{\sin(\omega t)}{\omega^3} - \frac{t}{\omega^2} + \right.$$

$$\left. + \frac{1}{\omega}\int_0^t dt' \, t' \sin(\omega t')\right] =$$

$$= \frac{F_0}{m\alpha}\left[\frac{\sin(\omega t)}{\omega^3} - \frac{t}{\omega^2} + \frac{1}{\omega}\left(\frac{\sin(\omega t)}{\omega^2} - \frac{t}{\omega}\cos(\omega t)\right)\right].$$

So, $$x(t) = \frac{F_0}{m\alpha\omega^3}\left[2\sin(\omega t) - \omega t(1+\cos(\omega t))\right], \quad 0 \leq t < \alpha. \quad (**)$$

Note that $x(0) = 0$ & $\dot{x}(0) = 0$ as requested.

For $t \geq \alpha$,

$$\ddot{x} = \frac{F_0}{m} \cos(\omega t), \quad \text{so that}$$

$$\begin{cases} \dot{x}(t) = \dfrac{F_0}{m\omega} \sin(\omega t) + b, \\ x(t) = -\dfrac{F_0}{m\omega^2} \cos(\omega t) + bt + a. \end{cases} \quad (***)$$

Initial conditions: use $(**)$ at $t = T$, yielding

$$\begin{cases} x(\alpha) = \dfrac{F_0}{m\alpha\omega^3} \left[2\sin(\omega\alpha) - \omega\alpha \left(1 + \cos(\omega\alpha)\right) \right], \\ \dot{x}(\alpha) = \dfrac{F_0}{m\alpha\omega^2} \left[\cos(\omega\alpha) - 1 + \omega\alpha \sin(\omega\alpha) \right]. \end{cases}$$

Then

$$b = \frac{F_0}{m\alpha\omega^2} \left(\cos(\omega\alpha) - 1 \right) \quad \text{and}$$

$$-\frac{F_0}{m\omega^2} \cos(\omega\alpha) + b\alpha + a =$$

$$= -\frac{F_0}{m\omega^2} \cos(\omega\alpha) + \frac{F_0}{m\omega^2} \left(\cos(\omega\alpha) - 1 \right) + a =$$

$$= \frac{2F_0}{m\alpha\omega^3} \sin(\omega\alpha) - \frac{F_0}{m\omega^2} \left(1 + \cos(\omega\alpha)\right), \quad \text{yielding}$$

$$a = \frac{F_0}{m\alpha\omega^3} \left[2\sin(\omega\alpha) - \omega\alpha \cos(\omega\alpha) \right].$$

a, b + $x(t)$ from $(***)$ constitute the desired solution.