HW#3 solutions

1. Start with the Lagrangian:

$$\mathcal{L} = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) + k\,\frac{e^{-ar}}{r}. \quad (*)$$

EoM for θ leads to $\ell = mr^2\dot{\theta} = \text{const}$.

EoM for r:

$$\begin{cases} \dfrac{\partial \mathcal{L}}{\partial \dot{r}} = m\dot{r}, \\ \dfrac{\partial \mathcal{L}}{\partial r} = mr\dot{\theta}^2 - k(1+ar)\,\dfrac{e^{-ar}}{r^2}. \end{cases}$$

$$\ddot{r} = r\underbrace{\dot{\theta}^2}_{\frac{\ell^2}{m^2r^4}} - \frac{k}{m}(1+ar)\,\frac{e^{-ar}}{r^2} =$$

$$= \frac{\ell^2}{m^2r^3} - \frac{k}{m}(1+ar)\,\frac{e^{-ar}}{r^2}. \quad (**)$$

Note that the effective potential

$$(*) \rightarrow V'(r) = -k\,\frac{e^{-ar}}{r} + \frac{\ell^2}{2mr^2}$$

$$\left.\frac{dV'(r)}{dr}\right|_{r_0} = 0 \quad \text{leads to}$$

$$k e^{-ar_0}\left(\frac{1}{r_0} + a\right) - \frac{\ell^2}{mr_0^2} = 0,$$

which is satisfied by $r_0 = \infty$ and by $\frac{km}{\ell^2}(1+ar_o)r_o = e^{ar_o}$ $\binom{*}{*}$

$\uparrow$

transcedental eq'n for r_o

It typically has $\emptyset$ or 2 solutions, and 1 solution as a special case

Note that if $a=0$, $\uparrow$ for larger values of a

$$\frac{km}{\ell^2} r_o = 1 \quad \Rightarrow \quad r_o = \frac{\ell^2}{mk} \quad \text{as expected.}$$

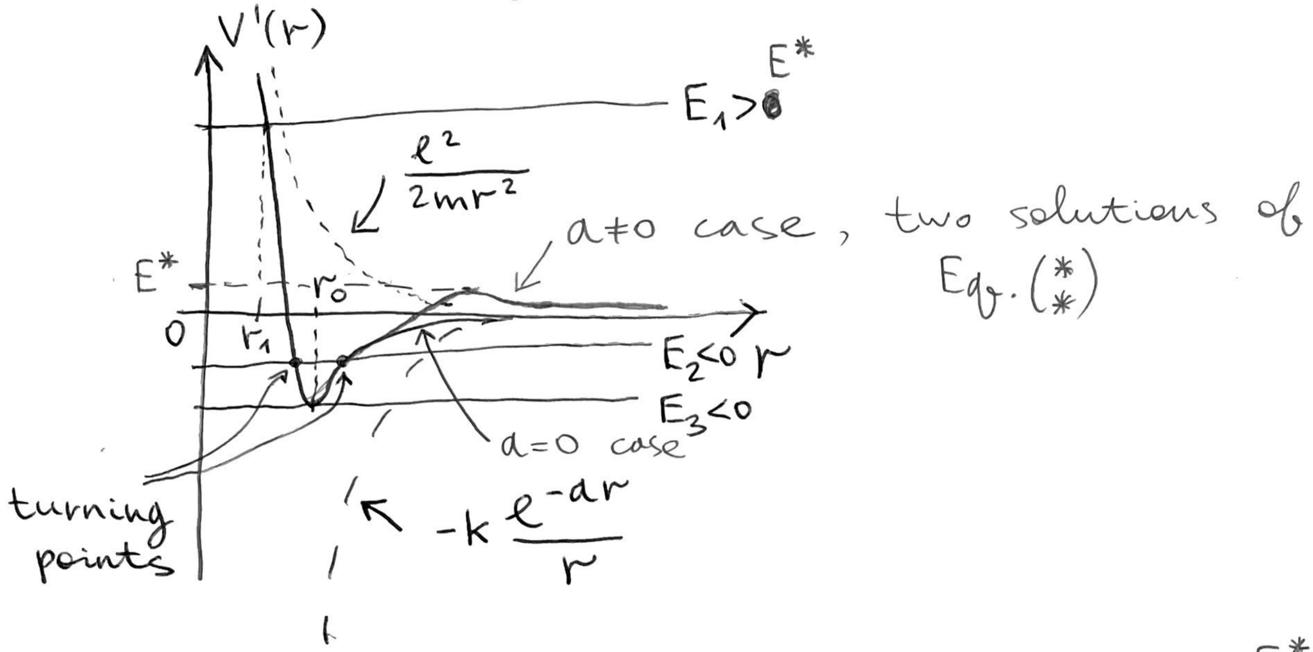
Note also that $V'(r_o) = \frac{\ell^2}{2mr_o^2} \frac{ar_o - 1}{ar_o + 1}$

Finally, Eq. (**) gives

$$\ddot{r}\Big|_{r_o} = \frac{\ell^2}{m^2 r_o^3} - \frac{k}{m}\cancel{(1+ar_o)}\frac{1}{r_o^2}\frac{\ell^2}{km}\frac{1}{\cancel{(1+ar_o)}r_o} =$$

$= 0$, as expected for a circular orbit.

Graphically,



Clearly, a particle with $E_1 > E^*$ has an unbounded trajectory: it comes in, turns around at $r = r_1$, and goes back out. The $r < r_1$ region is forbidden. For $E_2 < E^*$, the particle's orbit will be bounded between the two turning points shown in the Fig. above, but not necessarily closed. Finally, at $E_3 = V'(r_0)$, the particle will be in a stable circular orbit. The particle with $E_4 = E^*$ will be in an unstable circular orbit.

(2.) $\boxed{3.11}$

Gravitational forces:

$$U(r) = -\frac{k}{r},$$ leading to

$$\mathcal{L} = \frac{\mu}{2}(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{k}{r},$$ where

$\mu = \frac{m_1 m_2}{m_1 + m_2}$ is the reduced mass.

EoM is given by:

$$\mu \ddot{r} = \mu r \dot{\theta}^2 - \frac{k}{r^2}. \qquad (*)$$

For circular orbits, $\ddot{r}\big|_{r_0} = 0$, s.t.

$$r_0^3 = \frac{k}{\mu \dot{\theta}^2} = \frac{k\tau^2}{4\pi^2 \mu}$$

$\dot{\theta} = \frac{2\pi}{\tau}$ = const for circular orbits

When the particles are stopped, $\dot{\theta} = 0$ in Eq. (*), yielding

$$2\dot{r}\times \Big| \quad \ddot{r} = -\frac{k}{\mu r^2}$$ $\Leftarrow$ need to get $r(t)$ with $\begin{cases} r(0) = r_0, \\ \dot{r}(0) = 0 \end{cases}$

$$2\dot{r}\ddot{r} = -\frac{2k}{\mu r^2}\dot{r}, \quad \text{or}$$

$$\frac{d}{dt}(\dot{r}^2) = \frac{d}{dt}\left(\frac{2k}{\mu r}\right) \Rightarrow \dot{r}^2 = \frac{2k}{\mu r} + C_1$$

$$\dot{r}(0) = 0 \Rightarrow C_1 = -\frac{2k}{\mu r_0}.$$

Then $$\frac{dr}{dt} = \sqrt{\frac{2k}{\mu}}\sqrt{\underbrace{\frac{1}{r} - \frac{1}{r_0}}_{\frac{r_0 - r}{r r_0}}}.$$

Now, consider

$$\Delta t = \int_{r_0}^{0} dr\left(\frac{dt}{dr}\right) = \sqrt{\frac{\mu}{2k}}\int_{r_0}^{0} dr\sqrt{\frac{r r_0}{r_0 - r}}.$$

$$\int_{r_0}^{0} dr\sqrt{\frac{r r_0}{r_0 - r}} = r_0\int_1^0 du\sqrt{\frac{u r_0}{1-u}} = r_0^{3/2}\int_1^0 du\sqrt{\frac{u}{1-u}} =$$

$$\left\{\begin{array}{l} u = \frac{r}{r_0}, \\ du = \frac{dr}{r_0} \end{array}\right. \qquad \left\{\begin{array}{l} u = \sin^2 x, \\ du = 2\sin x\cos x\, dx \end{array}\right.$$

$$= r_0^{3/2}\int_{\frac{\pi}{2}}^{0} dx \times 2\sin x\cancel{\cos x}\,\frac{\sin x}{\cancel{\cos x}} = 2r_0^{3/2}\int_{\frac{\pi}{2}}^{0} dx\,\underbrace{\sin^2 x}_{\frac{1-\cos(2x)}{2}} =$$

$$= 2r_0^{3/2}\left[\frac{\pi}{4} - \underbrace{\frac{1}{2^2}\sin(2x)\Big|_{\frac{\pi}{2}}^{0}}_{\emptyset}\right] = 2r_0^{3/2}\frac{\pi}{4}.$$

Finally,

$$\Delta t = \sqrt{\frac{\mu}{2k}}\, r_0^{3/2}\,\frac{\pi}{2} = \sqrt{\frac{\mu}{2k}}\,\sqrt{\frac{k\tau^2}{4\cancel{\pi^2}\mu}}\,\frac{\cancel{\pi}}{2} = \frac{\tau}{4\sqrt{2}},$$

as desired

3. $\boxed{3.14}$

Orbit equation:

$$r^{-1} = \frac{mk}{\ell^2}\left(1 + \underbrace{e}_{=\sqrt{1+\frac{2E\ell^2}{mk^2}}} \cos\theta\right) \qquad (3.55)$$

$\swarrow$ $\theta' = 0$ for simplicity

(a) Circle has $e = 0 \Rightarrow r_c = \frac{\ell^2}{mk}$

Parabola has $e = 1 \Rightarrow r_p = \frac{\ell^2}{2mk} = \frac{r_c}{2}$.

$\theta = 0$ at perihelion

(b) In a circular orbit,

$$v_c = r\dot{\theta} = r\frac{\ell}{mr^2} = \frac{\ell}{mr}.$$

Recall that $\ell^2 = m\underset{\uparrow}{r}k$, (3.59)

circle radius

s.t.

$$v_c = \frac{\sqrt{mrk}}{mr} = \sqrt{\frac{k}{mr}}.$$

In a parabolic orbit,

$$v_p^2 = \dot{r}^2 + r^2\dot{\theta}^2.$$

From (3.55), $\dot{r} = \frac{\ell^2}{mk}\frac{d}{dt}\left(\frac{1}{1+\cos\theta}\right) \ominus$

$$\Leftrightarrow \frac{\ell^2}{mk}\frac{\sin\theta}{(1+\cos\theta)^2}\dot{\theta} = r\dot{\theta}\frac{\sin\theta}{1+\cos\theta}.$$

Then $\quad v_p^2 = r^2\dot{\theta}^2\left[\frac{\sin^2\theta}{(1+\cos\theta)^2}+1\right] =$

$$= r^2\dot{\theta}^2\,\frac{2+2\cos\theta}{(1+\cos\theta)^2} = \frac{2r^2\dot{\theta}^2}{1+\cos\theta} = \frac{2r^2\cancel{\ell^2}}{m^2r^4}\,\frac{mk}{\cancel{\ell^2}}\,r \Leftrightarrow$$

$$\begin{cases} \dot{\theta}^2 = \dfrac{\ell^2}{m^2r^4}, \\ \dfrac{1}{1+\cos\theta} = \dfrac{mk}{\ell^2}r \end{cases}$$

$$\Leftrightarrow \frac{2k}{mr}.$$

Finally, $\quad v_p = \sqrt{2}\sqrt{\frac{k}{mr}} = \underline{\underline{\sqrt{2}\,v_c}}.$

4. $\boxed{3.20}$

(a) Circular orbit:

$$T = \frac{2\pi}{\underbrace{\dot{\theta}}_{\omega}} \quad \& \quad \ell = mr^2\dot{\theta} \implies T = \frac{2\pi mr^2}{\ell}$$

Here, $f(r) = -\frac{k}{r^2} - m\underbrace{C}_{>0}r$

$$\Downarrow$$

$$|f(r)| = m\frac{v_c^2}{r} = mr\dot{\theta}^2 = mr\frac{\ell^2}{m^2r^4} = \frac{\ell^2}{mr^3}.$$

$$v_c = r\dot{\theta}$$

So, $\frac{k}{r_0^2} + mCr_0 = \frac{\ell^2}{mr_0^3}$, or

$$\ell = \sqrt{mr_0k + m^2Cr_0^4}.$$

Finally, $T = \frac{2\pi mr_0^2}{\sqrt{mr_0k + m^2Cr_0^4}} =$

$$= \frac{2\pi}{\sqrt{C + \frac{k}{mr_0^3}}}$$

Note that $\omega = \frac{2\pi}{T} = \sqrt{C + \frac{k}{mr_0^3}}$.

(b) use

$$\underbrace{u}_{=\frac{1}{r}} = u_0 + a\cos(\beta\theta) \qquad (3.45)$$

and

$$\beta^2 = 3 + \frac{r}{f}\frac{df}{dr}\bigg|_{r=r_0} \qquad (3.46)$$ for a circular orbit of radius r_0

Here, $f = -\frac{k}{r^2} - mCr$, which gives

$$\frac{df}{dr} = \frac{2k}{r^3} - mC .$$

Then $$\beta^2 = 3 + \frac{r_0}{\frac{k}{r_0^2} + mCr_0}\left(mC - \frac{2k}{r_0^3}\right) =$$

$$= \frac{\frac{3k}{r_0^2} + 3mCr_0 + mCr_0 - \frac{2k}{r_0^2}}{\frac{k}{r_0^2} + mCr_0} = \frac{4C + \frac{k}{mr_0^3}}{C + \frac{k}{mr_0^3}}$$

From (3.45) it is clear that

$$T_{osc} = \frac{T}{\beta} \underset{\substack{\uparrow \\ \text{use } T \text{ from} \\ \text{(a)}}}{=} \frac{2\pi/\sqrt{C + \frac{k}{mr_0^3}}}{\sqrt{4C + \frac{k}{mr_0^3}}\Big/\sqrt{C + \frac{k}{mr_0^3}}} = \frac{2\pi}{\sqrt{4C + \frac{k}{mr_0^3}}} .$$

Note that $\omega_{osc} = \frac{2\pi}{T_{osc}} = \sqrt{4C + \frac{k}{mr_0^3}}$.