HW #2 solutions

$\boxed{2.10}$

1. $y = at + bt^2 \Rightarrow \dot{y} = a + 2bt$

$V = -mgy$, $\underbrace{y > 0}_{\text{by construction}}$

Then $\mathcal{L} = \frac{m}{2}\dot{y}^2 + mgy =$

$$= \frac{m}{2}(a^2 + 4b^2t^2 + 4abt) + mg(at + bt^2) =$$

$$= \frac{ma^2}{2} + ma(2b+g)t + mb(2b+g)t^2, \text{ and}$$

$$\int_0^{t_0} dt\, \mathcal{L} = \frac{ma^2}{2} t_0 + \frac{ma}{2}(2b+g)t_0^2 + \frac{mb}{3}(2b+g)t_0^3.$$

Now, $y_0 = at_0 + bt_0^2$ implies that

$$a = \frac{y_0 - bt_0^2}{t_0}.$$

Then $$\frac{\partial}{\partial b}\left[\int_0^{t_0} dt\, \mathcal{L}\right] = mt_0 \cancel{\frac{y_0 - bt_0^2}{t_0}}(-t_0) + \frac{m}{2}(-t_0)(2b+g)t_0^2 +$$

$$+ mt_0^2 \cancel{\frac{y_0 - bt_0^2}{t_0}} + \frac{m}{3}(2b+g)t_0^3 + \frac{2mb}{3}t_0^3 = 0, \text{ or}$$

$$-\frac{m}{6}(2b+g) + \frac{2mb}{3} = 0,$$

$$\frac{b}{3} - \frac{g}{6} = 0 \Rightarrow b = +\frac{g}{2}.$$

Finally, $$a = \frac{y_0 - \frac{g}{2}\left(\frac{2y_0}{g}\right)}{\sqrt{\frac{2y_0}{g}}} = 0, \text{ as desired.}$$

2. $\boxed{2.12}$

Consider $\mathcal{L} = \mathcal{L}(\underbrace{q, \dot{q}, \ddot{q}}, t)$

for symplicity, focus on a single-particle case

and $I = \int_1^2 dt\, \mathcal{L}$.

Following Goldstein, consider

$$\underbrace{\delta I}_{\text{"} \frac{\partial I}{\partial \alpha} d\alpha} = \int_1^2 dt \left\{ \frac{\partial \mathcal{L}}{\partial q} \underbrace{\frac{\partial q}{\partial \alpha} d\alpha}_{\delta q} + \frac{\partial \mathcal{L}}{\partial \dot{q}} \frac{\partial \dot{q}}{\partial \alpha} d\alpha + \frac{\partial \mathcal{L}}{\partial \ddot{q}} \frac{\partial \ddot{q}}{\partial \alpha} d\alpha \right\} \boxed{=}$$

$$\int_1^2 dt\, \frac{\partial \mathcal{L}}{\partial \dot{q}} \frac{\partial^2 q}{\partial \alpha \partial t} = \cancel{\frac{\partial \mathcal{L}}{\partial \dot{q}} \frac{\partial q}{\partial \alpha} \Big|_1^2} - \int_1^2 dt\, \frac{\partial q}{\partial \alpha} \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right),$$

$$\int_1^2 dt\, \frac{\partial \mathcal{L}}{\partial \ddot{q}} \frac{\partial \ddot{q}}{\partial \alpha} = \int_1^2 dt\, \frac{\partial q}{\partial \alpha} \frac{d^2}{dt^2}\left(\frac{\partial \mathcal{L}}{\partial \ddot{q}}\right)$$

twice by parts, all boundary terms vanish

$$\boxed{=} \int_1^2 dt \left\{ \frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}} + \frac{d^2}{dt^2} \frac{\partial \mathcal{L}}{\partial \ddot{q}} \right\} \delta q = 0,$$ yielding

$$\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}} + \frac{d^2}{dt^2} \frac{\partial \mathcal{L}}{\partial \ddot{q}} = 0 \qquad (*)$$

Clearly, with n degrees of freedom we have:

$$\frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} + \frac{d^2}{dt^2}\frac{\partial \mathcal{L}}{\partial \ddot{q}_i} = 0, \qquad (**)$$

$$i = 1, \ldots, n$$

Now, plug

$$\mathcal{L} = -\frac{m}{2} q\ddot{q} - \frac{k}{2} q^2 \quad \text{into } (*):$$

$$\begin{cases} \dfrac{\partial \mathcal{L}}{\partial q} = -\dfrac{m}{2}\ddot{q} - kq, \\ \dfrac{\partial \mathcal{L}}{\partial \dot{q}} = 0, \\ \dfrac{\partial \mathcal{L}}{\partial \ddot{q}} = -\dfrac{m}{2} q \end{cases} \Rightarrow -\frac{m}{2}\ddot{q} - kq - \frac{m}{2}\ddot{q} = 0, \text{ or}$$

$$\underline{\underline{m\ddot{q} + kq = 0}}$$

EoM for a harmonic oscillator

So this $\mathcal{L}$ differs from a "normal" $\mathcal{L}$ for an oscillator by $\frac{d}{dt}F$.

$\Uparrow$ some f'n

$$\mathcal{L}_{sho} = \frac{m\dot{q}^2}{2} - \frac{kq^2}{2}$$

Indeed, $\mathcal{L} - \mathcal{L}_{sho} = -\frac{m}{2}q\ddot{q} - \frac{k}{2}q^2 - \frac{m}{2}\dot{q}^2 + \frac{k}{2}q^2 =$

$$= -\frac{m}{2}\frac{d}{dt}(q\dot{q}) \Rightarrow \underline{F = -\frac{m}{2}q\dot{q}}.$$

Thus EoMs obtained using $\mathcal{L}$ & $\mathcal{L}_{sho}$ will be the same.

(3.) $\boxed{2.16}$

$$\mathcal{L} = e^{\gamma t}\Big[\underbrace{\frac{m\dot{q}^2}{2} - \frac{kq^2}{2}}_{\text{sho}}\Big]$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \frac{\partial \mathcal{L}}{\partial q} = 0$$ then yields

$$\frac{d}{dt}\left(e^{\gamma t} m \dot{q}\right) + kq\, e^{\gamma t} = 0 \,,$$ or

$$m\ddot{q} + kq + \gamma m \dot{q} = 0 .$$

$$\Downarrow$$

$$\ddot{q} + \gamma \dot{q} + \frac{k}{m} q = 0$$ ← EoM for a damped harmonic oscillator

$$P = \frac{\partial \mathcal{L}}{\partial \dot{q}} = m\dot{q}\, e^{\gamma t} \,, \qquad \frac{dP}{dt} \neq 0$$ not conserved

↑ generalized momentum

$$E = e^{\gamma t}\left[\frac{m\dot{q}^2}{2} + \frac{kq^2}{2}\right], \qquad \frac{dE}{dt} = -\frac{\partial \mathcal{L}}{\partial t} \neq 0$$ not conserved

(must decrease with time)

↑ total energy

Next, use $s = e^{\gamma t} q$:

$$q = e^{-\gamma t} s \,, \qquad \dot{q} = -\gamma e^{-\gamma t} s + e^{-\gamma t}\dot{s} .$$

Then $\mathcal{L} = e^{\gamma t}\left[\frac{m}{2}(\dot{s}^2 + \gamma^2 s^2 - 2\gamma s\dot{s})e^{-2\gamma t} - \frac{k}{2} e^{-2\gamma t} s^2 \right] = e^{-\gamma t}\left[\frac{m}{2}\dot{s}^2 - m\gamma s\dot{s} + \frac{1}{2}(m\gamma^2 - k)s^2 \right]$.

$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{s}}\right) - \frac{\partial \mathcal{L}}{\partial s} = 0$ then yields

$$\frac{d}{dt}\left\{ e^{-\gamma t}[m\dot{s} - m\gamma s] \right\} + e^{-\gamma t} m\gamma\dot{s} - (m\gamma^2 - k)s e^{-\gamma t} = 0, \text{ or}$$

$$-\gamma[m\dot{s} - \cancel{m\gamma s}] + m\ddot{s} - \cancel{m\gamma\dot{s}} + \cancel{m\gamma\dot{s}} - \cancel{m\gamma^2 s} + ks = 0,$$

$$m\ddot{s} - \gamma m\dot{s} + ks = 0, \text{ or}$$

$$\ddot{s} - \gamma\dot{s} + \frac{k}{m}s = 0$$

↑ EoM for an "anti-damped" harmonic oscillator

$p = \frac{\partial \mathcal{L}}{\partial \dot{s}}$ not conserved since $\frac{\partial \mathcal{L}}{\partial s} \neq 0$.

E is also not conserved since $\frac{\partial \mathcal{L}}{\partial t} \neq 0$.

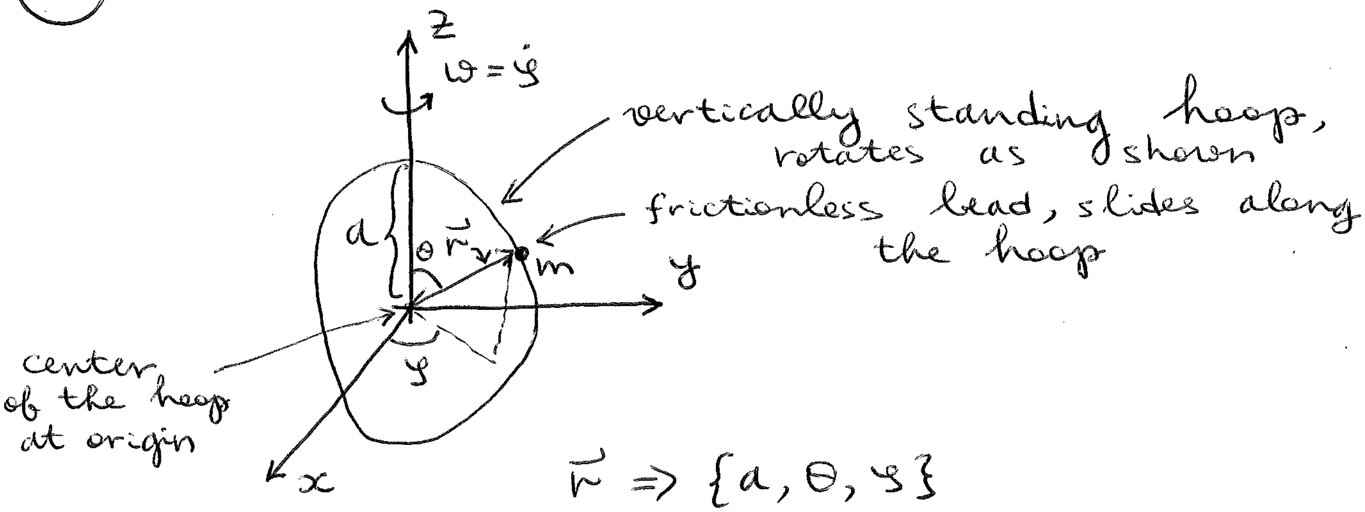
In fact, E must increase with time.

<u>Note:</u> it would have been more interesting to try

$$s = e^{\frac{\gamma}{2}t} q$$ as the explicit

time dependence would have canceled from $\mathcal{L}$. But this is not part of the problem ...

④ $\boxed{2.18}$



$$\vec{r} \Rightarrow \{a, \theta, \varphi\}$$

More explicitly,

$$\vec{r} = (a \sin\theta \cos\varphi,\ a \sin\theta \sin\varphi,\ a\cos\theta),$$

$$\vec{v} = \dot{\vec{r}} = a(\dot{\theta}\cos\theta\cos\varphi - \underbrace{\dot{\varphi}}_{=\omega}\sin\theta\sin\varphi,\ \dot{\theta}\cos\theta\sin\varphi +$$

$$+ \underbrace{\dot{\varphi}}_{=\omega}\sin\theta\cos\varphi,\ -\dot{\theta}\sin\theta).$$

Then $v^2 = a^2[\dot{\theta}^2\cos^2\theta\cos^2\varphi + \omega^2\sin^2\theta\sin^2\varphi -$

$- 2\omega\dot{\theta}\sin\theta\cancel{\cos\theta}\sin\varphi\cos\varphi + \dot{\theta}^2\cos^2\theta\sin^2\varphi +$

$+ \omega^2\sin^2\theta\cos^2\varphi + 2\omega\dot{\theta}\sin\theta\cancel{\cos\theta}\sin\varphi\cos\varphi + \dot{\theta}^2\sin^2\theta] =$

$= a^2[\dot{\theta}^2 + \omega^2\sin^2\theta].$

So, $\mathcal{L} = \frac{ma^2}{2}[\dot{\theta}^2 + \omega^2\sin^2\theta] - \underbrace{mga\cos\theta}_{0 \text{ when } \theta = \frac{\pi}{2}}$

Next, we employ

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot\theta}\right) - \frac{\partial \mathcal{L}}{\partial \theta} = 0 \quad \text{to obtain}$$

$$ma^2\ddot\theta = ma^2\omega^2 \sin\theta\cos\theta + mga\sin\theta .$$

Clearly, $p_\theta = \frac{\partial \mathcal{L}}{\partial \dot\theta}$ is not conserved but

$h = \dot\theta \frac{\partial \mathcal{L}}{\partial \dot\theta} - \mathcal{L}$ is: $\leftarrow$ since $\frac{\partial \mathcal{L}}{\partial t} = 0$

$$h = ma^2\dot\theta^2 - \frac{m}{2}a^2\dot\theta^2 - \frac{ma^2\omega^2}{2}\sin^2\theta + mga\cos\theta =$$

$$= \frac{ma^2}{2}\dot\theta^2 - \underbrace{\left[\frac{ma^2\omega^2}{2}\sin^2\theta - mga\cos\theta\right]}_{V_{eff}} = \text{const}.$$

We need to find stationary points of V_{eff}:

$$\frac{\partial V_{eff}}{\partial \theta} = -mga\sin\theta - ma^2\omega^2\sin\theta\cos\theta = 0, \text{ or}$$

$$\underbrace{\sin\theta}_{\theta = 0, \pi}\underbrace{(g + a\omega^2\cos\theta)}_{\cos\theta = -\frac{g}{a\omega^2} = -\frac{\omega_0^2}{\omega^2}} = 0 .$$

define $\omega_0^2 = \frac{g}{a}$
[units OK]

Clearly, if $\omega_0 > \omega$ there is no solution but for $\omega_0 \le \omega$ $\theta = \cos^{-1}\left(-\frac{\omega_0^2}{\omega^2}\right)$ becomes a valid alternative.

Stability analysis:

divide by $ma > 0$

Consider $\frac{\partial^2 V_{eff}}{\partial \theta^2} \Rightarrow \frac{\partial}{\partial \theta}\left[-\sin\theta\,(g + a\omega^2\cos\theta)\right] =$

$= -\cos\theta\,(g + a\omega^2\cos\theta) - \sin\theta \times a\omega^2(-\sin\theta) =$

$= -g\cos\theta - a\omega^2 \underbrace{\cos 2\theta}_{2\cos^2\theta - 1}.$

$\theta = 0$: $\quad -g - a\omega^2 < 0$, always unstable
[top of the hoop]

$\theta = \pi$: $\quad g - a\omega^2 > 0$ if $\omega^2 < \frac{g}{a} = \omega_0^2$ stable
[bottom of the hoop] $\quad \leq 0$ if $\omega^2 \geq \omega_0^2$ unstable

$\cos\theta = -\frac{\omega_0^2}{\omega^2}$: $\quad \frac{g\omega_0^2}{\omega^2} + a\omega^2\left[1 - 2\frac{\omega_0^4}{\omega^4}\right] = a\,\frac{\omega^4 - \omega_0^4}{\omega^2} \geq 0$
$[\omega \geq \omega_0]$ $\quad$ if $\omega^2 \geq \omega_0^2$, stable

So, the stable fixed point will switch from $\theta = \pi$ to $\theta = \cos^{-1}\left(-\frac{\omega_0^2}{\omega^2}\right)$ at $\omega = \omega_0$.

(5.) The system is translationally inv along the x-axis, yielding

$$\underbrace{v_1}_{\equiv|\vec{v}_1|} \sin\theta_1 = \underbrace{v_2}_{\equiv|\vec{v}_2|} \sin\theta_2 \, .$$

Moreover, the energy is conserved as well:

$$\frac{m v_1^2}{2} + U_1 = \frac{m v_2^2}{2} + U_2 \, , \quad \text{or}$$

$$v_2 = \sqrt{v_1^2 + \frac{2}{m}(U_1 - U_2)} \, .$$

Finally,

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_2}{v_1} = \sqrt{1 + \frac{2}{m v_1^2}(U_1 - U_2)} \, .$$

Note that $\theta_1 = \theta_2$ if $U_1 = U_2$, as expected.