HW #1 solutions (2022)

1.

3. Goldstein Ch. 1, Ex. 12 (escape velocity)

For the particle to escape from the Earth's gravitational field, it must, strictly speaking, be at ∞ distance from its surface:

$$V_{(r)} = -\underbrace{G}_{\text{gravit'l const}} \frac{m \overbrace{M}^{\text{Earth's mass}}}{\underbrace{R}_{\text{Earth's radius}} + \underbrace{r}_{\text{distance between particle \& Earth's surface}}}$$

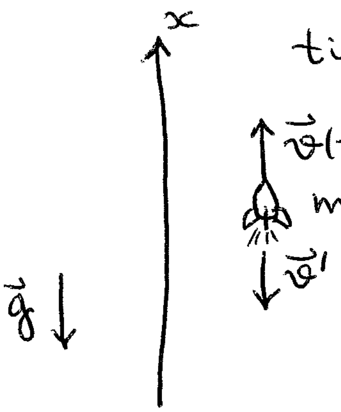
Clearly, $\lim_{r\to\infty} V(r) = 0 \Leftarrow$ "particle escaped"

At that moment, $T=0$ as well, since we are focusing on the minimum escape velocity. But then $\underbrace{E}_{\text{total energy}} = T+V = 0$ as well.

Since total energy is conserved, we have on the Earth surface:

$$\frac{m \overbrace{v_e^2}^{\text{escape velocity}}}{2} - G\frac{mM}{R} = 0 \Rightarrow v = \sqrt{\frac{2GM}{R}} \simeq 11.2\ \frac{km}{s}.$$

(2.) Goldstein Ch. 1, Ex. 13 (rockets)



time t:

$\vec{v}(t) \Leftarrow$ velocity

$m(t) \Leftarrow$ mass

$p(t) = m(t)\,v(t) \Leftarrow$ momentum

In a small interval Δt, the change in momentum is

$$\Delta p = F\Delta t = -g\,m(t)\,\Delta t$$

During Δt, the rocket releases Δm of gas at velocity v' wrt the rocket.

Thus
$$p(t+\Delta t) = \underbrace{\underbrace{(m(t)-\Delta m)}_{m(t+\Delta t)}\underbrace{(v(t)+\Delta v)}_{v(t+\Delta t)}}_{\text{rocket}} + \underbrace{\Delta m(v(t) - v')}_{\text{gas}}$$

Then
$$\Delta p = p(t+\Delta t) - p(t) \underset{\substack{\text{to } \theta(\Delta m),\\ \theta(\Delta v)}}{=} m(t)\,\Delta v - v'\Delta m = -g\,m(t)\,\Delta t,$$
or

$$\frac{\Delta v}{\Delta t} = -g + \frac{v'}{m(t)}\underbrace{\frac{\Delta m}{\Delta t}}$$

$$\gamma = \underbrace{\frac{m_0}{60}}_{\text{"} \alpha m_0,\ \alpha = \frac{1}{60}\ s^{-1}} s^{-1} \Leftarrow \text{rate of fuel expenditure}$$

(m_0 ← initial mass of the rocket)

Note that
$$\underbrace{\frac{\Delta v}{\Delta t}}_{>0} = -\underbrace{\gamma}_{\substack{>0\\ \text{by def'n}}}\underbrace{\frac{\Delta v}{\Delta m}}_{<0}\ [v \nearrow \text{ as } m \searrow]$$

Then $-\gamma \frac{\Delta v}{\Delta m} = -g + \frac{v'}{m}\gamma$, or,

in the $\Delta t \to 0$, $\Delta m \to 0$, $\Delta v \to 0$ limit:

$$dv = \frac{g}{\gamma} dm - v' \frac{dm}{m}.$$

This yields

$$v(m) = \frac{g}{\gamma}(m - m_0) - v' \log\left(\frac{m}{m_0}\right). \qquad (*)$$

$v(m_0) = 0$

We need to find $\frac{m_0}{m}$ in terms of v_e, i.e. invert Eq. (*). Using $m \ll m_0$, we obtain:

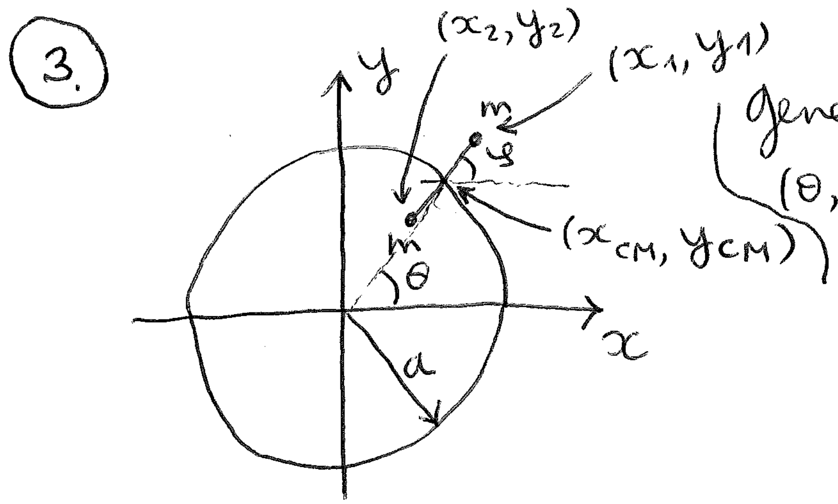
$$v_e = \frac{g}{\alpha}\left(\frac{m}{m_0} - 1\right) - |\vec{v}'| \log\left(\frac{m}{m_0}\right) \simeq$$

$$\simeq -\frac{g}{\alpha} + |\vec{v}'| \log\left(\frac{m_0}{m}\right). \qquad (**)$$

Plugging in the numbers and inverting Eq. (**), we obtain:

$$\frac{m_0}{m} \simeq 293, \quad \text{almost } \underline{\underline{300}}$$

3.



Generalized coordinates: (θ, φ)

$$(x_{CM}, y_{CM}) = (a\cos\theta, a\sin\theta)$$

Relative to the CM,

$$\begin{cases} (x_1^{rel}, y_1^{rel}) = \frac{\ell}{2}(\cos\varphi, \sin\varphi), \\ (x_2^{rel}, y_2^{rel}) = -\frac{\ell}{2}(\cos\varphi, \sin\varphi). \end{cases}$$

Then
$$\begin{cases} (x_1, y_1) = (a\cos\theta + \frac{\ell}{2}\cos\varphi, a\sin\theta + \frac{\ell}{2}\sin\varphi), \\ (x_2, y_2) = (a\cos\theta - \frac{\ell}{2}\cos\varphi, a\sin\theta - \frac{\ell}{2}\sin\varphi). \end{cases}$$

Combine the two expressions and drop 1,2 subscripts:

$$(\dot{x}, \dot{y}) = (-a\sin\theta\,\dot{\theta} \mp \frac{\ell}{2}\sin\varphi\,\dot{\varphi}, \; a\cos\theta\,\dot{\theta} \pm \frac{\ell}{2}\cos\varphi\,\dot{\varphi}).$$

Then
$$v_{1,2}^2 = \dot{x}^2 + \dot{y}^2 = a^2\dot{\theta}^2 + \frac{\ell^2}{4}\dot{\varphi}^2 \pm a\ell\dot{\theta}\dot{\varphi}(\underbrace{\cos\theta\cos\varphi + \sin\theta\sin\varphi}_{\cos(\theta-\varphi)}).$$

Finally, $T = \frac{m}{2}(v_1^2 + v_2^2) =$

$$= m\left(a^2\dot{\theta}^2 + \frac{\ell^2}{4}\dot{\varphi}^2\right).$$

4\. Use $\mathcal{L} = T - V$

For m_1, we have:

$$\begin{cases} T_1 = \frac{m_1}{2}(l_1\dot{\theta}_1)^2, \\ V_1 = -m_1 g l_1 \cos\theta_1 \end{cases}$$

$\quad = -m_1 g y_1$

For m_2, we have:

$$\begin{cases} x_2 = l_1 \sin\theta_1 + l_2 \sin\theta_2, \\ y_2 = l_1\cos\theta_1 + l_2\cos\theta_2. \end{cases}$$

↑ Cartesian coords of m_2

Then $T_2 = \frac{m_2}{2}(\dot{x}_2^2 + \dot{y}_2^2) = \frac{m_2}{2}\Big[\underline{l_1^2\cos^2\theta_1\dot{\theta}_1^2} +$

$+ l_2^2\cos^2\theta_2\,\dot{\theta}_2^2 + 2l_1l_2\cos\theta_1\cos\theta_2\,\dot{\theta}_1\dot{\theta}_2 +$

$+ \underline{l_1^2\sin^2\theta_1\,\dot{\theta}_1^2} + l_2^2\sin^2\theta_2\,\dot{\theta}_2^2 + 2l_1l_2\sin\theta_1\sin\theta_2\Big] =$

$= \frac{m_2}{2}\Big[l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1l_2\cos(\theta_1-\theta_2)\,\dot{\theta}_1\dot{\theta}_2\Big].$

Finally, $V_2 = -m_2 g y_2$.

Then $\mathcal{L} = T_1 + T_2 - V_1 - V_2 =$

$= \frac{m_1+m_2}{2}l_1^2\dot{\theta}_1^2 + \frac{m_2}{2}l_2\dot{\theta}_2^2 + m_2 l_1 l_2\cos(\theta_1-\theta_2)\dot{\theta}_1\dot{\theta}_2 +$

$$+ (m_1+m_2)gl_1\cos\theta_1 + m_2 g l_2\cos\theta_2.$$

5. again, use $\mathcal{L} = T - V$:

$$m_1: \begin{cases} T_1 = \dfrac{m_1 \dot{x}^2}{2}, \\ V_1 = 0 \end{cases}$$

$$m_2: \begin{cases} x_2 = x + l \sin\theta, \\ y_2 = l\cos\theta. \end{cases}$$

Then $T_2 = \frac{m_2}{2}(\dot{x}_2^2 + \dot{y}_2^2) = \frac{m_2}{2}(\dot{x}^2 + l^2 \cos^2\theta\,\dot{\theta}^2 +$

$+ l^2 \sin^2\theta\,\dot{\theta}^2 + 2\dot{x} l \cos\theta\,\dot{\theta}) =$

$= \frac{m_2}{2}(\dot{x}^2 + l^2\dot{\theta}^2 + 2l\cos\theta\,\dot{x}\dot{\theta}),$

$V_2 = -m_2 g y_2 = -m_2 g l \cos\theta.$

Finally,

$$\mathcal{L} = \frac{m_1 + m_2}{2}\dot{x}^2 + \frac{m_2}{2} l^2 \dot{\theta}^2 + m_2 l \cos\theta\,\dot{x}\dot{\theta} +$$
$$+ m_2 g l \cos\theta.$$