Final solutions (2022)

① EoM:

$$\begin{cases} m_1 \ddot{x}_1 = m_1 g - f_1, \\ m_2 \ddot{x}_2 = m_2 g - f_2, \\ m_3 \ddot{x}_3 = m_3 g - f_3. \end{cases} \quad (*)$$

f_i are tension forces

Note that $\begin{cases} f_2 = f_3, \\ f_1 = f_2 + f_3 \end{cases}$

Moreover, $\begin{cases} x_1 + \pi r + x_0 = l_1, \\ (x_2 - x_0) + (x_3 - x_0) + \pi r = l_2, \end{cases}$

where x_0 is the x-position of the center of the lower pulley.

This leads to

$$2x_1 + x_2 + x_3 = 2l_1 + l_2 - 3\pi r, \text{ or}$$

$$2\ddot{x}_1 + \ddot{x}_2 + \ddot{x}_3 = \underline{\underline{0}}.$$

Next,

$$(*) \Rightarrow \begin{cases} m_1\ddot{x}_1 - m_2\ddot{x}_2 - m_3\ddot{x}_3 = \\ = (m_1 - m_2 - m_3) g \underbrace{- f_1 + f_2 + f_3}_{=0}, \\ \\ m_2\ddot{x}_2 - m_3\ddot{x}_3 = (m_2 - m_3) g. \end{cases}$$

$\uparrow f_2 = f_3$

Now, $\begin{cases} \ddot{x}_2 = \dfrac{(m_2-m_3)g + m_3\ddot{x}_3}{m_2}, \\[2ex] \ddot{x}_1 = \dfrac{m_2\ddot{x}_2 + m_3\ddot{x}_3 + (m_1-m_2-m_3)g}{m_1} = \end{cases}$

$$= \frac{(m_2-m_3)g + 2m_3\ddot{x}_3 + (m_1-m_2-m_3)g}{m_1}.$$

Therefore,

$$\frac{2(m_2-m_3)g + 4m_3\ddot{x}_3 + 2(m_1-m_2-m_3)g}{m_1} +$$

$$+ \frac{(m_2-m_3)g + m_3\ddot{x}_3}{m_2} + \ddot{x}_3 = 0, \text{ or}$$

$$2m_2(m_2-m_3)g + 4m_2m_3\ddot{x}_3 + 2m_2(m_1-m_2-m_3)g +$$

$$+ m_1(m_2-m_3)g + m_1m_3\ddot{x}_3 \underbrace{+\ddot{x}_3}_{m_1m_2} = 0,$$

$$\ddot{x}_3 = \frac{(4m_2m_3 - 3m_1m_2 + m_1m_3)}{\underbrace{4m_2m_3 + m_1m_3 + m_1m_2}_{\equiv A}} g.$$

Sanity check: if $m_2 = m_3 = \frac{1}{2}m_1$,

$4m_2m_3 - 6m_2m_3 + 2m_2m_3 = 0$ as expected
(the weights are balanced)

Finally, $x_3(t) = x_{3,0} + \dfrac{Agt^2}{2}$]

Note that as $m_1 \to \infty$, $A \to \dfrac{-3m_2+m_3}{m_2+m_3}$,
may be <0 or >0 depending on m_2/m_3

Next, $\ddot{x}_2 = \frac{(m_2 - m_3)}{m_2} g + \frac{m_3}{m_2} A g =$

$$= \underbrace{\frac{m_2 - (1-A) m_3}{m_2}}_{\equiv B} g .$$

$\rightarrow$ note that $B = 0$ for $m_1 = 2m_2 = 2m_3$

$$x_2(t) = x_{2,0} + \frac{B g t^2}{2} .]$$

$$\ddot{x}_1 = - \frac{\ddot{x}_2 + \ddot{x}_3}{2} = \underbrace{- \frac{A+B}{2}}_{\equiv C} g , \text{ or}$$

$\rightarrow$ note that $C = 0$ for $m_1 = 2m_2 = 2m_3$

$$x_1(t) = x_{1,0} + \frac{C g t^2}{2} .$$

$\boxed{\text{QED}}$

(2.) The spheres will move away from the rotation axis until they run into the stops at the ends of the rod.

Conservation of angular momentum:

$$L = (I_0 + 2mR_0^2)\,\omega_0 = (I_0 + 2mR^2)\,\omega = \text{const}$$

$t=0$ $t>0$

R = distance between the axis of rotation and the sphere's center at t

The corresponding kinetic energy is

$$T_{rot} = \frac{1}{2} L\omega = \frac{1}{2}(I_0 + 2mR^2)\,\omega^2 .$$

The maximum distance $R_m = \frac{L}{2} - R_0$,

so that

$$\omega_1 = \frac{(I_0 + 2mR_0^2)\,\omega_0}{I_0 + 2m\left(\frac{L}{2} - R_0\right)^2} < \omega_0 \quad \text{since}$$

$$R_m = \frac{L}{2} - R_0 > R_0 .$$

But then $\Delta T_{rot} = \frac{L}{2}(\omega_1 - \omega_0) < 0$.

It's not surprising since T_{rot} does not include the kinetic energy of the moving spheres, which becomes heat once the spheres hit the stops inelastically. With completely elastic

collisions one would observe perfect periodic oscillations between R_0 and R_m. In practice, there will be a decaying oscillation until ΔT_{rot} is entirely lost.
due to dissipation

(3.) The generalized coordinates are (θ, φ).

The kinetic energy is

$$T = \frac{1}{2} m b^2 \dot{\theta}^2 + \frac{1}{2} m b^2 \sin^2\theta \, \dot{\varphi}^2 .$$

b is the radius of the circle for θ

$b \sin\varphi$ is the radius of the circle for φ

The potential energy is

$$\mathcal{U} = -mgb \cos\theta$$

$\mathcal{U} = 0$ if $\theta = \frac{\pi}{2}$, as desired

The generalized momenta are

$$\begin{cases} p_\theta = \dfrac{\partial \mathcal{L}}{\partial \dot{\theta}} = m b^2 \dot{\theta}, \\ \mathcal{L} = T - \mathcal{U} \\ p_\varphi = \dfrac{\partial \mathcal{L}}{\partial \dot{\varphi}} = m b^2 \sin^2\theta \, \dot{\varphi} \end{cases}$$

The Hamiltonian is given by

$$H = T + \mathcal{U} = \frac{p_\theta^2}{2mb^2} + \frac{p_\varphi^2}{2mb^2 \sin^2\theta} - mgb\cos\theta .$$

Finally, Hamilton's EoM are:

$$\begin{cases} \dot{\theta} = \dfrac{\partial H}{\partial p_\theta} = \dfrac{p_\theta}{mb^2}, \\ \dot{\varphi} = \dfrac{\partial H}{\partial p_\varphi} = \dfrac{p_\varphi}{mb^2 \sin^2\theta}, \\ \dot{p}_\theta = -\dfrac{\partial H}{\partial \theta} = \dfrac{p_\varphi^2 \cos\theta}{mb^2 \sin^3\theta} - mgb\sin\theta, \\ \dot{p}_\varphi = -\dfrac{\partial H}{\partial \varphi} = 0 \end{cases}$$

$\hookrightarrow$ $p_\varphi = \mathrm{const}(t)$ $\Rightarrow$ φ is a cyclic coordinate.

4. Poisson brackets (PB) are defined by

$$[g, h] = \sum_k \left(\frac{\partial g}{\partial q_k} \frac{\partial h}{\partial p_k} - \frac{\partial g}{\partial p_k} \frac{\partial h}{\partial q_k} \right).$$

(a) $$\frac{dg}{dt} = \frac{\partial g}{\partial t} + \frac{\partial g}{\partial q_k} \underbrace{\dot{q}_k}_{\frac{\partial H}{\partial p_k}} + \frac{\partial g}{\partial p_k} \underbrace{\dot{p}_k}_{-\frac{\partial H}{\partial q_k}} =$$

sum over k implied

$$= \frac{\partial g}{\partial t} + \frac{\partial g}{\partial q_k} \frac{\partial H}{\partial p_k} - \frac{\partial g}{\partial p_k} \frac{\partial H}{\partial q_k} = \frac{\partial g}{\partial t} + [g, H].$$

(b) Using (a), we immediately obtain

$$\begin{cases} \dot{p}_j = [p_j, H], \\ \dot{q}_j = [q_j, H] \end{cases} \leftarrow \begin{cases} \frac{\partial p_j}{\partial t} = 0, \\ \frac{\partial q_j}{\partial t} = 0 \end{cases}$$

(c) $$[p_i, p_j] = \cancel{\frac{\partial p_i}{\partial q_k}} \frac{\partial p_j}{\partial p_k} - \frac{\partial p_i}{\partial p_k} \cancel{\frac{\partial p_j}{\partial q_k}} = 0.$$

$[q_i, q_j] = 0$ as well

$$[p_i, q_j] = \cancel{\frac{\partial p_i}{\partial q_k} \frac{\partial q_j}{\partial p_k}} - \underbrace{\frac{\partial p_i}{\partial p_k}}_{\delta_{ik}} \underbrace{\frac{\partial q_j}{\partial q_k}}_{\delta_{jk}} = -\delta_{ij}$$

(d) Finally, if $g = \text{const}(t)$:

$$\frac{dg}{dt} = 0 = \frac{\partial g}{\partial t} + [g, H].$$

Thus, we need $\begin{cases} \dfrac{\partial g}{\partial t} = 0, \\ [g, H] = 0. \end{cases}$

(5.) (a) Use $\left.\frac{dv}{dr}\right|_{r_0} = 0$:

$$4\epsilon\left[-\frac{12}{r}\left(\frac{\sigma}{r}\right)^{12} + \frac{6}{r}\left(\frac{\sigma}{r}\right)^{6}\right]\Bigg|_{r_0} = 0 \text{ , or}$$

$$-2\left(\frac{\sigma}{r_0}\right)^6 + 1 = 0 \implies \underline{\underline{r_0 = 2^{1/6}\sigma.}}$$

(b) Expand $v(r)$ around r_0:

$$v(r) = \underbrace{v(r_0)}_{\text{const}} + \underbrace{\left.\left(\frac{dv}{dr}\right)\right|_{r_0}}_{=0}(r-r_0) + \frac{1}{2}\left.\left(\frac{d^2v}{dr^2}\right)\right|_{r_0}(r-r_0)^2 + \dots$$

$$\approx v(r_0) + \frac{1}{2}\left.\left(\frac{d^2v}{dr^2}\right)\right|_{r_0}(r-r_0)^2 .$$

The force is given by

Hooke's law

$$F(r) = -\frac{dv}{dr} = -\underbrace{\left.\left(\frac{d^2v}{dr^2}\right)\right|_{r_0}}_{=k = \text{ force constant}}(r-r_0) .$$

Then $\omega = \sqrt{\frac{k}{m}} = \sqrt{\left.\left(\frac{d^2v}{dr^2}\right)\right|_{r_0}\frac{1}{m}}$, where

$m = \frac{m_1 m_2}{m_1+m_2}$ is the reduced mass.

Finally,

$$\left.\frac{d^2 v}{dr^2}\right|_{r_0} = 4\epsilon \left[\frac{12\times 13}{r^2}\left(\frac{\sigma}{r}\right)^{12} - \frac{6\times 7}{r^2}\left(\frac{\sigma}{r}\right)^{6} \right]\Bigg|_{r_0} =$$

$$= 4\epsilon \left[\frac{39}{r_0^2} - \frac{21}{r_0^2} \right] = 4\epsilon \frac{18}{r_0^2} = \frac{72\,\epsilon}{2^{1/3}\sigma^2}.$$

$$\left(\frac{\sigma}{r_0}\right)^6 = \frac{1}{2}$$

Thus, $\omega = 2^{1/3}\cdot 6 \sqrt{\frac{\epsilon}{m}}\,\frac{1}{\sigma}$.

The average separation $\langle r \rangle$ will increase with the amplitude of the oscillations because $v(r)$ is not symmetric about r_0: time spent at $r > r_0$ is greater than time spent at $r < r_0$.