Midterm solutions (2021)

(1.) Generalized coordinates : $\{r, \theta\}$

Since the particle is sliding on the surface of the hemisphere, the constraint equation is given by:

$$f(r,\theta) = r - a = 0 .$$

$$\begin{cases} T = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2), \\ V = mgr\cos\theta \end{cases} \Rightarrow \mathcal{L} = T - V = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - mgr\cos\theta$$

↑ choose $V = 0$ if $\theta = \frac{\pi}{2}$.

Constraint forces:

$$\begin{cases} Q_r = \lambda \frac{\partial f}{\partial r} = \lambda, \\ Q_\theta = \lambda \frac{\partial f}{\partial \theta} = 0. \end{cases}$$

EoM:

$$\begin{cases} \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) - \frac{\partial \mathcal{L}}{\partial r} = \lambda, \\ \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) - \frac{\partial \mathcal{L}}{\partial \theta} = 0. \end{cases}$$

We have:

$$\begin{cases} m\ddot{r} - mr\dot{\theta}^2 + mg\cos\theta - \lambda = 0, \\ mr^2\ddot{\theta} - mgr\sin\theta + 2mr\dot{r}\dot{\theta} = 0. \end{cases}$$

However, the constraint implies:

$$\begin{cases} r = a, \\ \dot{r} = 0, \\ \ddot{r} = 0 \end{cases}$$

This yields

$$\begin{cases} ma\dot{\theta}^2 - mg\cos\theta + \lambda = 0, \\ ma^2\ddot{\theta} - mga\sin\theta = 0. \end{cases}$$

$$\rightarrow \ddot{\theta} = \frac{g}{a}\sin\theta$$

Using $\ddot{\theta} = \dot{\theta}\frac{d\dot{\theta}}{d\theta}$, we obtain:

$$\int \dot{\theta}\, d\dot{\theta} = \frac{g}{a}\int d\theta \sin\theta \text{ , or}$$

$$\frac{\dot{\theta}^2}{2} = -\frac{g}{a}\cos\theta + \underbrace{\frac{g}{a}}_{\theta = 0,\ \dot{\theta} = 0\ @\ t = 0}$$

Now, $\lambda = mg\cos\theta - ma\left[-\frac{2g}{a}\cos\theta + \frac{2g}{a}\right] =$

$= 3mg\cos\theta - 2mg = mg(3\cos\theta - 2).$

Now recall that

$\lambda = Q_r$, the radial constraint force.

The particle will fall off the hemisphere when $\lambda = 0$, giving:

$$mg(3\cos\theta_0 - 2) = 0, \text{ or}$$

$$\theta_0 = \cos^{-1}\left(\frac{2}{3}\right).$$

The answers are:

note that $Q_r = mg$ when $\theta = 0$

(a) $\begin{cases} Q_r = mg(3\cos\theta - 2) \\ Q_\theta = 0 \end{cases}$

(b) $\theta_0 = \cos^{-1}\left(\frac{2}{3}\right)$.

(2.) Recall that

$$E = \underbrace{\frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2)}_{T} + \underbrace{V(r)}_{V}, \text{ where}$$

$$\dot{\theta}^2 = \frac{\ell^2}{m^2 r^4}.$$

Here, $V(r) = -\frac{k}{r}$, with $k = G \underset{\text{Earth mass}}{M_e} \underset{\text{satellite mass}}{m_s}$

For a circular orbit,

$$R = \frac{\ell^2}{mk}$$

m: reduced mass,

$$\frac{1}{m} = \frac{1}{M_e} + \frac{1}{m_s}$$

Then $V(R) = -\frac{k^2 m}{\ell^2}$ and

$$T = \frac{m}{2} R^2 \frac{\ell^2}{m^2 R^4} = \frac{mk^2}{2\ell^2}.$$

Note that $\underline{\underline{T = -\frac{V(R)}{2}}}$. (*)

(a) Firing of the engine does not change $V(R)$, but T becomes:

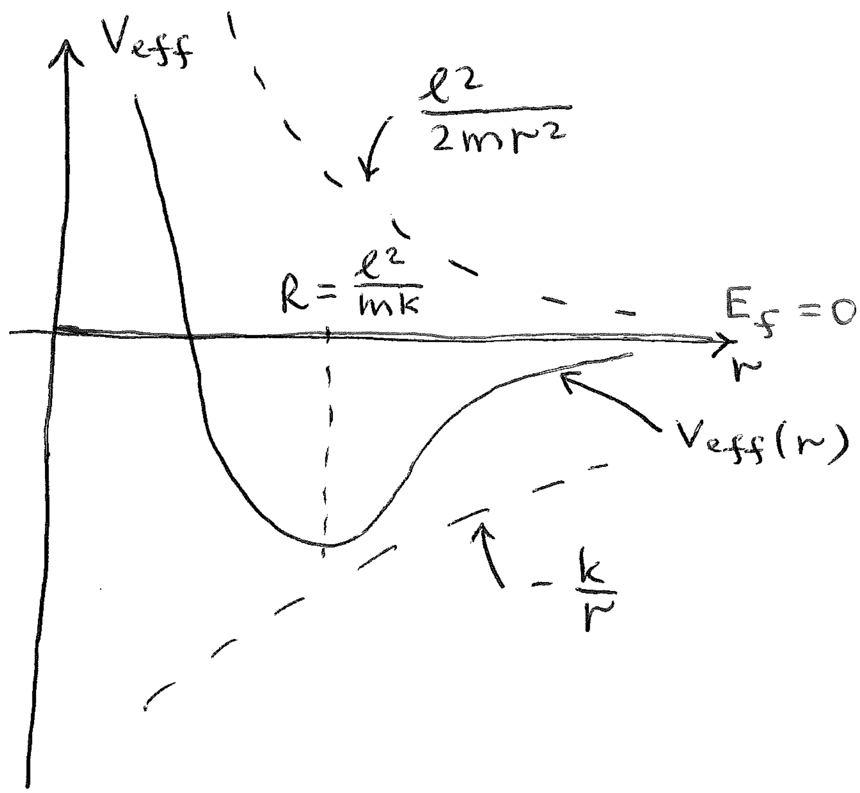
$$T_f = \frac{m}{2}\underbrace{v^2}_{\substack{v = R\dot{\theta},\\ \text{old angular velocity}}} + \frac{m}{2}\underbrace{v^2}_{\substack{\text{new radial}\\ \text{velocity}}} = 2T$$

So, $$E_f = 2T + V(R) = \overbrace{-V(R)}^{\text{from } (*)} + V(R) = 0 .$$

The angular momentum does not change since the satellite is boosted radially: $\ell_f = \ell$.

$E_f = 0$ means that the new orbit is parabolic and the satellite will leave the Earth's orbit.

(b) As in class,



The satellite will start on the parabolic orbit after firing its engine, starting from $r = R$ and increasing r as time goes on.