Midterm solutions

(1.) Particle 1:

$$\mathcal{L}_1 = \frac{m_1}{2}(\dot r^2 + r^2\dot\varphi^2 + \dot z^2) - m_1 g z,$$

where φ is the azimuthal angle and $r^2 = x^2 + y^2$.

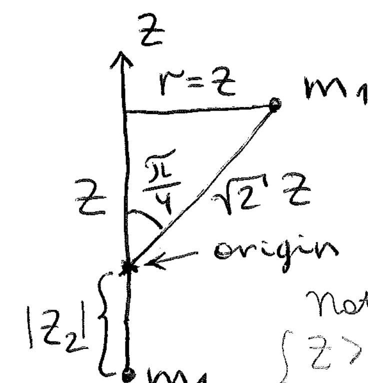
Cone surface constraint:

$$\tan\theta = \frac{r}{z} = 1 \quad \text{for} \quad \theta = \frac{\pi}{4}, \text{ s.t.}$$

$r = z$ and thus

$$\mathcal{L}_1 = \frac{m_1}{2}(2\dot z^2 + z^2\dot\varphi^2) - m_1 g z$$

Particle 2:



$$\mathcal{L}_2 = \frac{m_2}{2}\dot z_2^2 - m_2 g z_2$$

Note that $\begin{cases} z > 0, \\ z_2 < 0 \end{cases}$

However,

$$\underbrace{-z_2}_{>0} + \sqrt{2}\, z = L, \quad \text{or}$$

$$\begin{cases} z_2 = -(L - \sqrt{2}\, z), \\ \dot z_2 = +\sqrt{2}\,\dot z. \end{cases}$$

Finally, $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 = \frac{m_1}{2}(2\dot z^2 + z^2\dot\varphi^2) - m_1 g z +$

$$+ m_2 \dot z^2 - m_2\sqrt{2}\, g z$$

← + const which is OK to discard

The generalized coords are (z, φ).

(2.) (a) Recall that $\vec{\ell} = \vec{r} \times \vec{p}$.

Then

$$\frac{d\vec{\ell}}{dt} = \underbrace{\frac{d\vec{r}}{dt} \times \vec{p}}_{\vec{v} \times m\vec{v} = 0} + \underbrace{\vec{r} \times \frac{d\vec{p}}{dt}}_{} = 0 .$$

$\frac{d\vec{p}}{dt} = \vec{F} = -\vec{\nabla} U =$

$= -\hat{r}\frac{dU}{dr}$ for a central force field, yielding

$\vec{r} \times \left(-\hat{r}\frac{dU}{dr}\right) = 0$

Thus $\vec{\ell} =$ const

(b) $\vec{\ell} = \vec{r} \times \vec{p} \Rightarrow \vec{\ell} \perp \vec{r}, \ \vec{\ell} \perp \vec{p}$.

Since $\vec{\ell} =$ const, $\vec{r} \perp \vec{\ell}$ for any $t \Rightarrow$

$\Rightarrow$ the motion is in a plane $\perp \vec{\ell}$.

(c) $E = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) + U(r)$.

However, $\underset{\substack{\| \\ |\vec{\ell}|}}{\ell} = mr^2\dot{\theta} \Rightarrow \dot{\theta} = \frac{\ell}{mr^2}$.

Then $E = \frac{m}{2}\dot{r}^2 + \frac{m}{2}r^2\frac{\ell^2}{m^2r^4} + U(r) =$

$$= \frac{m}{2}\dot{r}^2 + \underbrace{\left(\frac{\ell^2}{2mr^2} + U(r)\right)}_{\equiv\, U_{eff}(r)} .$$

(d) For circular orbits, $r = r_0 = \text{const}$, s.t.

$E = \mathcal{U}_{eff}(r_0)$. $\Leftarrow$ E must intersect $\mathcal{U}_{eff}(r)$ somewhere for a circular orbit to exist

This implies that

$$\left.\frac{d\mathcal{U}_{eff}}{dr}\right|_{r_0} = 0, \quad \text{since}$$

$$\underbrace{-\left.\frac{d\mathcal{U}}{dr}\right|_{r_0}}_{\text{radial force}} = \underbrace{-m r_0 \dot{\theta}^2}_{\text{radial acceleration}} = -m r_0 \frac{\ell^2}{m^2 r_0^4} = -\frac{\ell^2}{m r_0^3}.$$

Then

$$\underbrace{\left.\frac{d\mathcal{U}}{dr}\right|_{r_0} - \frac{\ell^2}{m r_0^3} = 0}_{\text{circular orbit equation}} = \underline{\underline{\left.\frac{d\mathcal{U}_{eff}}{dr}\right|_{r_0}}}.$$

(e) Consider

$$\mathcal{U}_{eff}(r) \simeq \mathcal{U}_{eff}(r_0) + \cancel{\left.\frac{d\mathcal{U}_{eff}}{dr}\right|_{r_0}(r - r_0)} + \frac{1}{2}\left.\frac{d^2\mathcal{U}_{eff}}{dr^2}\right|_{r_0} \times (r - r_0)^2.$$

expansion around r_0 — see above

If $\left.\frac{d^2\mathcal{U}_{eff}}{dr^2}\right|_{r_0} > 0$, $\mathcal{U}_{eff}(r_0)$ is a local minimum, and the circular orbit is stable (i.e., the force which appears when r deviates slightly from r_0 restores the orbit).

(f) $\vec{F} = -\frac{dU}{dr}\hat{r} = -\left(\frac{b}{r^2} - \frac{c}{r^4}\right)\hat{r}$.

For circular orbits,

$$\left.\frac{dU_{eff}}{dr}\right|_{r_0} = 0 \Rightarrow \frac{b}{r_0^2} - \frac{c}{r_0^4} - \frac{\ell^2}{mr_0^3} = 0, \text{ or}$$

$$br_0^2 - \frac{\ell^2}{m}r_0 - c = 0. \qquad (*)$$

Then $r_0 = \dfrac{\ell^2/m \pm \sqrt{\ell^4/m^2 + 4bc}}{2b}$.

Take the "+" root,

$$r_0 = \frac{\ell^2/m + \sqrt{\ell^4/m^2 + 4bc}}{2b} \geq \sqrt{\frac{c}{b}}$$

i.e., $r_0 \to \sqrt{\frac{c}{b}}$ as $\ell \to 0$ and $r_0 \uparrow$ as $\ell \uparrow$.

Stability analysis:

$$\left.\frac{d^2U_{eff}}{dr^2}\right|_{r_0} = -\frac{2b}{r_0^3} + \frac{4c}{r_0^5} + \frac{3\ell^2}{mr_0^4}$$

But $br_0^2 = \frac{\ell^2}{m}r_0 + c$, or

$\underset{(*)}{\Uparrow}$

$$\frac{3\ell^2}{m}r_0 = 3br_0^2 - 3c, \text{ yielding}$$

$$\left.\frac{d^2 U_{eff}}{dr^2}\right|_{r_0} = -\frac{2b}{r_0^3} + \frac{4c}{r_0^5} + \frac{3b}{r_0^3} - \frac{3c}{r_0^5} =$$

$$= \frac{b}{r_0^3} + \frac{c}{r_0^5} > 0.$$

If $c \to 0$, $r_0 = \frac{\ell^2}{bm}$, as expected.

Moreover, $\left.\frac{d^2 U_{eff}}{dr^2}\right|_{r_0} = \frac{b}{r_0^3}$ is always satisfied

This can be summarized by saying that if $c=0$, for any $r_0 \geq 0$, there is a value of ℓ for which this r_0 is a radius of a stable circular orbit.

If $c \neq 0$, the force has to be attractive: for the circular orbit to exist

$$\frac{b}{r_0^2} - \frac{c}{r_0^4} > 0 \implies r_0^2 b - c > 0, \text{ or}$$

$$r_0^2 > \frac{c}{b}, \text{ same as above}$$

Of course, the force is always attractive if $c=0$.