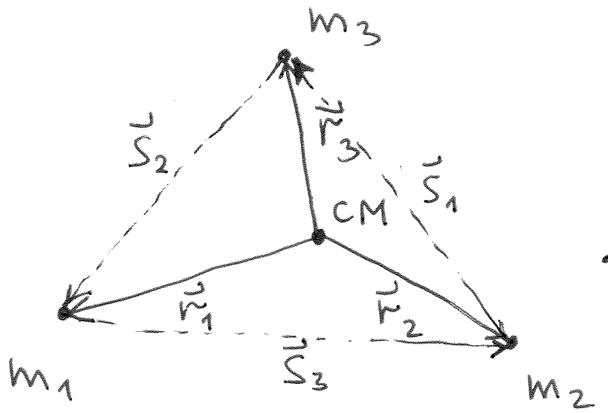
Lecture 9

The 3-body problem

(Newtonian dynamics)

$$\begin{cases} \ddot{\vec{r}}_1 = -Gm_2 \dfrac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|^3} - Gm_3 \dfrac{\vec{r}_1 - \vec{r}_3}{|\vec{r}_1 - \vec{r}_3|^3}, \\ \ddot{\vec{r}}_2 = \ldots, \qquad \ddot{\vec{r}}_3 = \ldots \end{cases}$$



Introduce

$$\begin{cases} \vec{S}_1 = \vec{r}_3 - \vec{r}_2, \\ \vec{S}_2 = \vec{r}_1 - \vec{r}_3, \\ \vec{S}_3 = \vec{r}_2 - \vec{r}_1 \end{cases}$$

Clearly, $\vec{S}_1 + \vec{S}_2 + \vec{S}_3 = 0$.

Now, consider

$$\ddot{\vec{S}}_2 = \ddot{\vec{r}}_1 - \ddot{\vec{r}}_3 = +Gm_2 \frac{\vec{S}_3}{S_3^3} - Gm_3 \frac{\vec{S}_2}{S_2^3} - Gm_1 \frac{\vec{S}_2}{S_2^3} \textcircled{+}$$

$$\textcircled{+} Gm_2 \frac{\vec{S}_1}{S_1^3} \diamond\!\!=$$

$$\Nearrow$$

$$\ddot{\vec{r}}_3 = -Gm_1 \frac{\vec{r}_3 - \vec{r}_1}{|\vec{r}_3 - \vec{r}_1|^3} - Gm_2 \frac{\vec{r}_3 - \vec{r}_2}{|\vec{r}_3 - \vec{r}_2|^3}$$

$$\diamond\!\!= -G(\underbrace{m_3 + m_1 + m_2}_{\equiv m}) \frac{\vec{S}_2}{S_2^3} + Gm_2 \underbrace{\left(\frac{\vec{S}_3}{S_3^3} + \frac{\vec{S}_2}{S_2^3} + \frac{\vec{S}_1}{S_1^3} \right)}_{\frac{\vec{G}}{G}} \textcircled{=}$$

$$\textcircled{=} -mG \frac{\vec{S}_2}{S_2^3} + m_2 \vec{G}$$

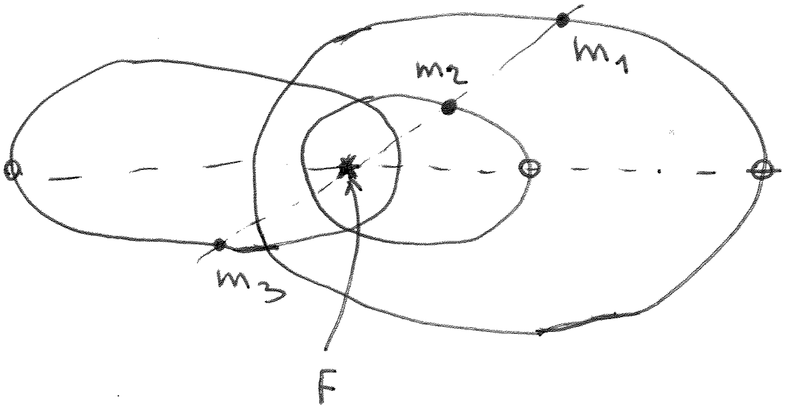
Same for $\ddot{\vec{S}}_1$ & $\ddot{\vec{S}}_3$ ...

The 3 coupled eq's for $\ddot{\vec{S}}_i$ cannot be solved in general, but some solutions are known.

(Euler's solution)

① $\vec{r}_1, \vec{r}_2, \vec{r}_3$ are collinear $\Rightarrow$ $\vec{S}_1, \vec{S}_2, \vec{S}_3, \vec{G}$ are all collinear with $\vec{r}_i$'s.

There's a bound state in which the 3 masses are always collinear:



3 confocal ellipses, same period τ

o ≡ aphelion position

(Lagrange's solution)

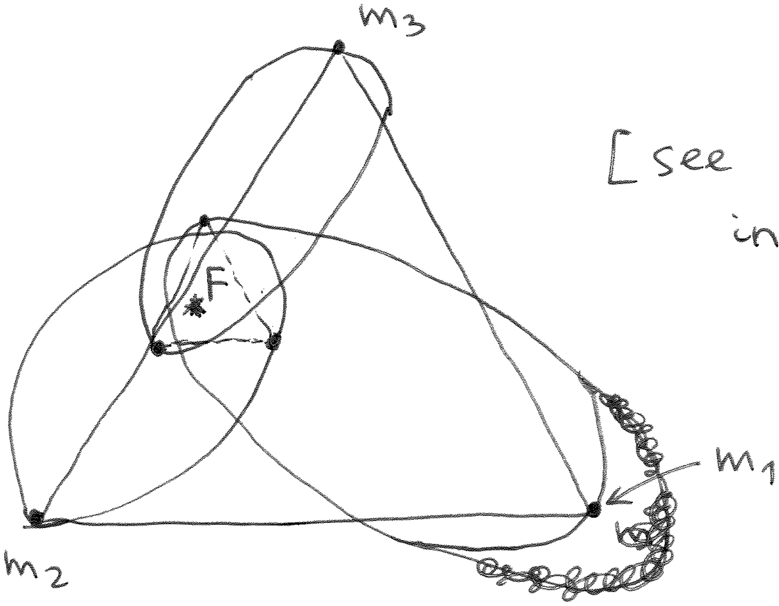
② $\vec{G}=0$ solution $\Rightarrow$ if $S_1 = S_2 = S_3 = S$, $\vec{G}=0$ always holds: equilateral triangle in the "$m_1 - m_2 - m_3$" Fig.

Then $\ddot{\vec{S}}_i = - m G \dfrac{\vec{S}_i}{S_i^3}$, $i=1, 2, 3$

and the eq's are decoupled

as $t \uparrow$, the eq's remain decoupled, so that m_1, m_2, m_3 are always vertices on an equilateral triangle, but it changes its orientation & size:

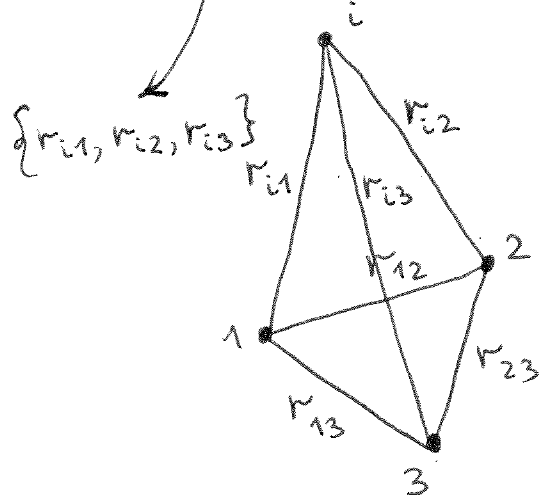
bound state with elliptical orbits



[see Fig. 3.29 in Goldstein]

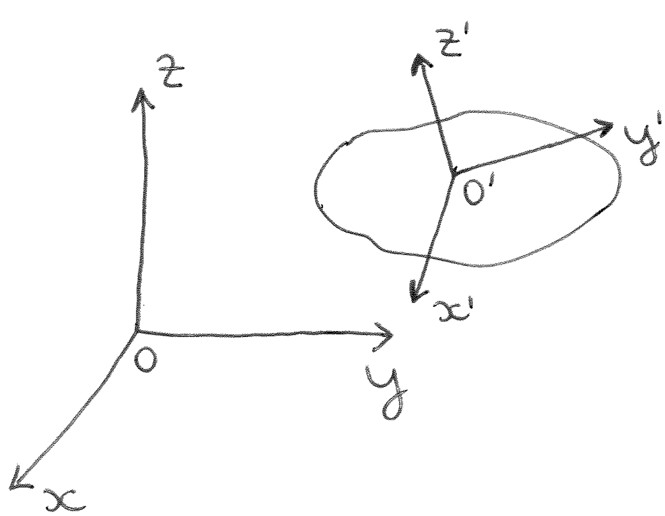
Kinematics of rigid body motion

Consider a rigid body with N particles (3N DoF) $\Rightarrow \frac{N(N-1)}{2}$ constraints of the form $r_{ij} = C_{ij}$, $\forall i,j$. These constraints are not all independent. How many are then? Any point can be described through its distances to 3 non-collinear points, which are described by 9 DoF:



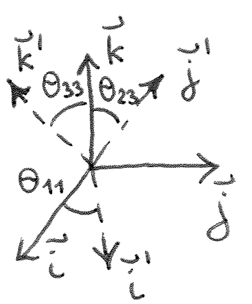
However, 3 constraints: $\begin{cases} r_{12} = C_{12}, \\ r_{23} = C_{23}, \\ r_{13} = C_{13} \end{cases}$ reduce the # of DoFs to $\underline{\underline{6}}$.

So, we need to assign 6 generalized coords to a rigid body. Consider a body frame & a lab frame:



3 DoFs to specify the origin of the body frame (O') + 3 DoFs to specify its orientation wrt lab frame.

Consider direction cosines:



$$\begin{cases} \cos\theta_{11} = \vec{i}'\cdot\vec{i} = \vec{i}'\cdot\vec{i}, \\ \cos\theta_{12} = \vec{i}'\cdot\vec{j} = \vec{j}\cdot\vec{i}', \\ \text{etc.} \end{cases}$$

Further,
$$\begin{cases} \vec{i}' = \cos\theta_{11}\,\vec{i} + \cos\theta_{12}\,\vec{j} + \cos\theta_{13}\,\vec{k}, \\ \vec{j}' = \cos\theta_{21}\,\vec{i} + \cos\theta_{22}\,\vec{j} + \cos\theta_{23}\,\vec{k}, \\ \vec{k}' = \cos\theta_{31}\,\vec{i} + \cos\theta_{32}\,\vec{j} + \cos\theta_{33}\,\vec{k}. \end{cases}$$

Then
$$\begin{cases} x' = \vec{r}\cdot\vec{i}' = \cos\theta_{11}\,\overbrace{(\vec{r}\cdot\vec{i})}^{x} + \cos\theta_{12}\,\overbrace{(\vec{r}\cdot\vec{j})}^{y} + \\ \quad + \cos\theta_{13}\,\underbrace{(\vec{r}\cdot\vec{k})}_{z} = \cos\theta_{11}\,x + \cos\theta_{12}\,y + \cos\theta_{13}\,z, \\ y' = \cos\theta_{21}\,x + \cos\theta_{22}\,y + \cos\theta_{23}\,z, \\ z' = \cos\theta_{31}\,x + \cos\theta_{32}\,y + \cos\theta_{33}\,z. \end{cases}$$

Note that $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} = x'\vec{i}' + y'\vec{j}' + z'\vec{k}'$, so that the direction cosines can be used to express both $\{x', y', z'\}$ through $\{x, y, z\}$ and vice versa.

Clearly, 9 direction cosines completely determine the transformation between the 2 coord systems. But they are not all indep. since

$$\begin{cases} \vec{i}\cdot\vec{j} = \vec{j}\cdot\vec{k} = \vec{i}\cdot\vec{k} = 0, \\ \vec{i}\cdot\vec{i} = \vec{j}\cdot\vec{j} = \vec{k}\cdot\vec{k} = 1 \end{cases}$$

Same for $\vec{i}', \vec{j}', \vec{k}'$

For example,

$$\vec{i}\cdot\vec{j} = \left(\cos\theta_{11}\,\vec{i}' + \cos\theta_{21}\,\vec{j}' + \cos\theta_{31}\,\vec{k}'\right) \times$$

$$\times \left(\cos\theta_{12}\,\vec{i}' + \cos\theta_{22}\,\vec{j}' + \cos\theta_{32}\,\vec{k}'\right) =$$

$$= \cos\theta_{11}\cos\theta_{12} + \cos\theta_{21}\cos\theta_{22} + \cos\theta_{31}\cos\theta_{32} =$$

$$= \sum_{l=1,2,3} \cos\theta_{l1}\cos\theta_{l2} = 0 .$$

More generally,

$$\sum_{l=1}^{3} \cos\theta_{lm'}\cos\theta_{lm} = \delta_{m'm} \qquad (*)$$

$$m, m' = 1, 2, 3$$

Since $(*)$ is inv wrt $m \Leftrightarrow m'$, we have 6 indep. eq's $\Rightarrow$ only 3 DoFs are independent.

Now change notation:

$$\begin{cases} x \to x_1, \quad y \to x_2, \quad z \to x_3 \\ a_{ij} = \cos\theta_{ij} \end{cases}$$

Then

$$\begin{cases} x_1' = a_{11} x_1 + a_{12} x_2 + a_{13} x_3, \\ x_2' = a_{21} x_1 + a_{22} x_2 + a_{23} x_3, \\ x_3' = a_{31} x_1 + a_{32} x_2 + a_{33} x_3, \end{cases} \qquad (*)$$

or

$$(**) \qquad x_i' = \underbrace{a_{ij} x_j}_{\text{sum over } j \text{ implied}}, \qquad i = 1, 2, 3 \qquad]$$

a special case of a general linear transform

Note that $\underbrace{x_i' x_i'}_{\overset{\shortparallel}{a_{ij} a_{ik} x_j x_k}} = x_i x_i$ vector magnitude

$$\Downarrow$$

$$(***) \qquad \underbrace{a_{ij} a_{ik}}_{\cos\theta_{ij} \cos\theta_{ik}} = \delta_{jk} \qquad [\delta_{jk} x_j x_k = x_k x_k]$$

Eqs. (**) & (***) together define an orthogonal transform'n.

$$A = \begin{pmatrix} a_{11} & a_{12} & .. \\ a_{21} & a_{22} & .. \\ .. & .. & .. \end{pmatrix}$$ is the transformation matrix

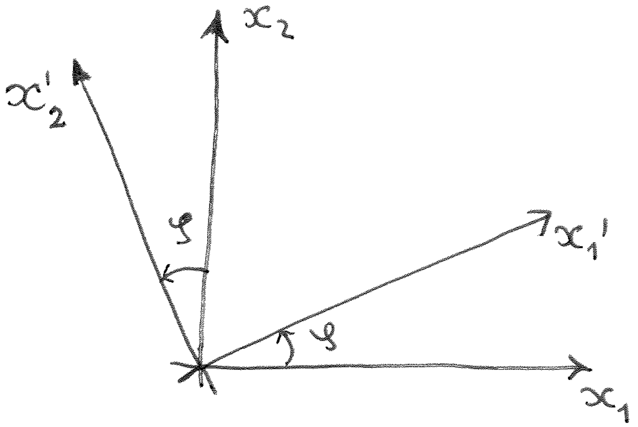
For example, for a 2D rotation in a plane

$$A = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Since $a_{ij} a_{ik} = \delta_{jk}$ for $\begin{cases} j=1, \; k=1 \\ j=2, \; k=2 \\ j=1, \; k=2 \end{cases}$,

we have 3 constraints $\Rightarrow$ only 1 DoF. is indep.

Use rotation angle φ:



$$\begin{cases} x_1' = x_1 \cos\varphi + x_2 \sin\varphi, \\ x_2' = -x_1 \sin\varphi + x_2 \cos\varphi \end{cases}$$

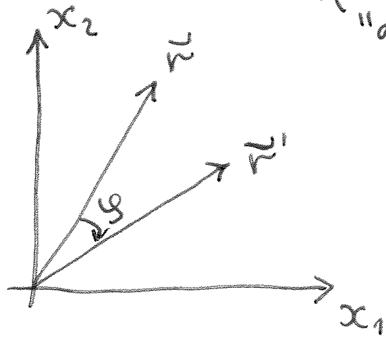
$$A = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{cases} a_{1\overset{\overset{j}{\downarrow}}{1}} a_{1\overset{\overset{k}{\downarrow}}{1}} + a_{21} a_{21} = \cos^2\varphi + \sin^2\varphi = 1, \\ a_{12} a_{12} + a_{22} a_{22} = \sin^2\varphi + \cos^2\varphi = 1, \\ a_{1\underset{\underset{j}{\uparrow}}{1}} a_{1\underset{\underset{k}{\uparrow}}{2}} + a_{21} a_{22} = \cos\varphi \cdot \sin\varphi - \sin\varphi \cdot \cos\varphi = \underline{\underline{0}}. \end{cases}$$

Note that $$\underset{\substack{\shortparallel\\ \begin{pmatrix}x'\\y'\\z'\end{pmatrix}}}{\vec r\,'} = A\,\underset{\substack{\shortparallel\\ \begin{pmatrix}x\\y\\z\end{pmatrix}}}{\vec r}$$ ⇐ either transforms $\vec r$ into a new coord. system OR rotates $\vec r$ into $\vec r\,'$ while the coord. system is fixed:

↑ "passive"

↑ "active"



Properties of the transformation matrix

$$\vec r\,'' = A\vec r\,' = \underbrace{AB}_{\substack{\shortparallel\\ C}}\vec r \neq BA\vec r \quad \text{in general}$$

However, $(AB)C = A(BC)$.

Inverse: $\vec r\,' = A\vec r \Rightarrow \vec r = A^{-1}\vec r\,' = A^{-1}(A\vec r)$

⇙

$A^{-1}A = \mathbb{1}$

Likewise, $\vec r\,' = A(A^{-1}\vec r\,') \Rightarrow \underline{\underline{AA^{-1} = \mathbb{1}}}$

Thus A & A^{-1} commute. The unit matrix $\mathbb{1}$ is s.t. $\vec r = \mathbb{1}\vec r$ & $\mathbb{1}A = A\mathbb{1} = A$

$$A^{-1}A = \mathbb{1} \quad \text{gives} \quad \underbrace{a'_{ij}}_{\substack{\text{elements}\\ \text{of } A^{-1}}}\underbrace{a_{jk}}_{\substack{\text{elements}\\ \text{of } A}} = \delta_{ik}$$

Likewise, $AA^{-1} = \mathbb{1}$ gives $a_{ki}a'_{ij} = \delta_{kj}$

Next, consider

$$(\underbrace{a_{kl}\, a_{ki}}_{\delta_{li}})\, a'_{ij} = a'_{lj}$$

On the other hand,

$$a_{kl}\,(\underbrace{a_{ki}\, a'_{ij}}_{\delta_{kj}}) = a_{jl} .$$

So, $a'_{lj} = a_{jl} \Rightarrow A^{-1} = \underbrace{\tilde{A}}_{\text{matrix transpose}}$

So, $A\tilde{A} = \tilde{A}A = \mathbb{1}$.

Note that $a_{ij} x_j = x_j \underbrace{(\tilde{A})_{ji}}_{``a_{ij}}$, s.t.

$$Ax = x\tilde{A}$$

A is symmetric when $a_{ij} = a_{ji}$,
antisymmetric when $a_{ij} = -a_{ji}$.

Now, consider $\overset{\text{vectors}}{G = \underbrace{A}_{\text{matrix}} F}$ (*)

Transform (*) to a new coord. system:

$$BG = BAF = \underbrace{(BAB^{-1})}_{\text{acts on } BF \text{ (i.e., } F \text{ in the new system) to produce } BG \text{ (i.e., } G \text{ in the new system)}} BF$$

So, $A' = BAB^{-1}$ is a similarity transformation for A

Determinants: $|AB| = |A| \times |B|$, implying

that $\underbrace{|\tilde{A}|}_{|\tilde{A}| = |A|} \times |A| = |A|^2 = 1 \Rightarrow |A| = \pm 1$ for an orthogonal matrix

Finally, $A'B = BA$ gives

$|A'| * |B| = |B| * |A|$, or

$|A'| = |A| \Leftarrow$ det is inv under a similarity transform'n