Lecture 7

Orbit equation

So far, we have discussed $r(t)$ & $\theta(t)$. However, often $r(\theta)$ is of even greater interest.

Note that $\ell = mr^2\dot{\theta}$ implies

$$\ell\, dt = mr^2 d\theta, \text{ or}$$

$$\frac{d}{dt} = \frac{\ell}{mr^2}\frac{d}{d\theta}.$$

Now, $m\ddot{r} = f(r) + \frac{\ell^2}{mr^3}$ gives

$$\frac{\ell}{r^2}\frac{d}{d\theta}\left(\frac{\ell}{mr^2}\frac{dr}{d\theta}\right) = f(r) + \frac{\ell^2}{mr^3}.$$

Introduce $u = \frac{1}{r} \Rightarrow \frac{1}{r^2}\frac{dr}{d\theta} = -\frac{du}{d\theta}$.

Then $\frac{\ell}{r^2}\frac{d}{d\theta}\left(-\frac{\ell}{m}\frac{du}{d\theta}\right) = -\frac{\ell^2 u^2}{m}\frac{d^2u}{d\theta^2}$, and

$$-\frac{\ell^2 u^2}{m}\frac{d^2 u}{d\theta^2} = f\left(\frac{1}{u}\right) + \frac{\ell^2 u^3}{m},$$

$$\frac{d^2u}{d\theta^2} + u = -\frac{m}{\ell^2 u^2} f\left(\frac{1}{u}\right) = -\frac{m}{\ell^2}\frac{d}{du}V\left(\frac{1}{u}\right).$$

$$f(r) = -\frac{dV(r)}{dr} = -\frac{dV\left(\frac{1}{u}\right)}{\underbrace{d\left(\frac{1}{u}\right)}_{-u^{-2}du}} = u^2\frac{dV\left(\frac{1}{u}\right)}{du}$$

Note that the eq'n is invariant wrt $\theta \to -\theta$. When θ is chosen s.t. $\theta = 0$ at the turning point, the motion will be symmetric around the $\theta = 0$ axis.

Moreover,

$$\underbrace{dt}_{\frac{mr^2}{\ell}d\theta} = \frac{dr}{\sqrt{\frac{2}{m}\left(E - V - \frac{\ell^2}{2mr^2}\right)}}$$ gives

$$d\theta = \frac{\ell\, dr}{mr^2\sqrt{\cdots}} = \frac{dr}{r^2\sqrt{\frac{2mE}{\ell^2} - \frac{2mV}{\ell^2} - \frac{1}{r^2}}}.$$

Then

$$\theta = \int_{r_0}^{r} \frac{dr'}{r'^2\sqrt{\frac{2mE}{\ell^2} - \frac{2mV}{\ell^2} - \frac{1}{r'^2}}} + \theta_0, \text{ or}$$

$$\Downarrow \; u = 1/r$$

$$\theta = \theta_0 - \int_{u_0}^{u} du' \frac{1}{\sqrt{\frac{2mE}{\ell^2} - \frac{2mV}{\ell^2} - u'^2}}. \qquad (*)$$

Eq'n $(*)$ does not yield closed-form solutions in general.

$V = a r^{n+1}$ yields elementary solutions for $n = 1, -2, -3$

Conditions for closed orbits

Recall that a circular orbit is observed if $V'(r)$ has a min or a max @ $r=r_0$ & if $E=V'(r_0)$:

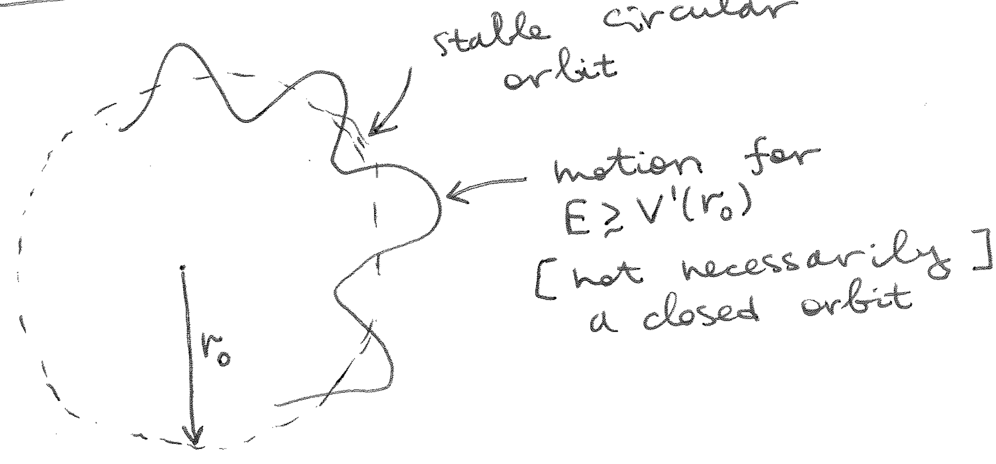
$$f'(r_0)=0 \implies \underbrace{f(r_0)}_{\text{must be attractive, or negative}} = -\frac{\ell^2}{mr_0^3}$$

Further, $E=V(r_0)+\dfrac{\ell^2}{2mr_0^2}$.

If $V'(r_0)$ is a min, the orbit with $E \gtrsim V'(r_0)$ is still bounded, but no longer circular:

$\Uparrow$ the circular orbit is <u>stable</u>

[if $V'(r_0)$ is a max, the circular orbit is <u>unstable</u>]



<u>Stability condition:</u>

$$\left.\frac{\partial^2 V'}{\partial r^2}\right|_{r=r_0} = -\left.\frac{\partial f}{\partial r}\right|_{r=r_0} + \frac{3\ell^2}{mr_0^4} \underbrace{>0}_{V' \text{ concave up}}$$

Next,

$$\left.\frac{\partial f}{\partial r}\right|_{r=r_0} < \frac{3}{r_0} \underbrace{\frac{\ell^2}{m r_0^3}}_{-f(r_0)} = -\frac{3 f(r_0)}{r_0} \quad \text{, or}$$

$$\left.\frac{\partial \log f}{\partial \log r}\right|_{r=r_0} > -3$$

note that $\frac{f(r_0)}{r_0}$ is negative

If $f = -\underbrace{k}_{>0} r^n$,

$$-k n r_0^{n-1} < 3 k r_0^{n-1} \text{, or}$$

$$n > -3$$ ← condition for the circular orbit to be stable

Now, consider

$$f(r) = -\frac{k}{r^{\underbrace{3-\beta^2}_{-n}}} \Rightarrow n = \beta^2 - 3 > -3 \quad \text{if } \beta \neq 0$$

If deviations from the circular orbit are small, one can do 2nd order expansion of $V'(r)$ & study the resulting oscillatory motion. It turns out that if β is a rational number (i.e. $\beta = \frac{p}{q}$, where $p, q \in \mathbb{Z}$), the orbits will be closed $\Rightarrow$ after q complete revolutions the orbit starts retracing itself.

What about non-perturbative analysis?
[J. Bertrand, 1873]

The orbits are closed only for $\beta^2 = 1$
$\Big(\underbrace{f(r) = -\frac{k}{r^2}}_{\text{inverse square law}} \Big)$ & $\beta^2 = 4$ $\Big(\underbrace{f(r) = -kr}_{\text{Hooke's law}} \Big)$.

Since closed orbits are observed in astrophysics for a wide variety of perturbations & initial conditions, the force law must have the inverse-square form. (!)