Lecture 5

Conservation laws and symmetries

EoM are $\underset{\substack{\uparrow \\ \text{\# generalized coords}}}{n}$ 2nd order eq's, can be solved, resulting in $2n$ const of integration, to be determined by the initial conditions. However, often insights can be provided w/out obtaining the complete solution of the EoM.

Indeed, we can focus on the first integrals of the EoM:

$$f(q_1, q_2, \ldots; \dot{q}_1, \dot{q}_2, \ldots; t) = \text{const}$$

For ex., consider a system of particles under $V(q_1, q_2, \ldots)$:

Recall that $$\frac{\partial \mathcal{L}}{\partial \dot{x}_i} = \frac{\partial T}{\partial \dot{x}_i} = m_i \dot{x}_i = \underbrace{p_{i,x}}_{x \text{ component of linear momentum } \vec{p}_i}$$

Correspondingly, the generalized momentum is defined as $$p_j = \frac{\partial \mathcal{L}}{\partial \dot{q}_j}$$

If $\mathcal{L}$ does not depend on q_j, (such a coord is called cyclic)

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_j} - \frac{\partial \mathcal{L}}{\partial q_j} = 0 \Rightarrow \frac{d}{dt}\underbrace{\frac{\partial \mathcal{L}}{\partial \dot{q}_j}}_{p_j} = 0, \text{ or}$$

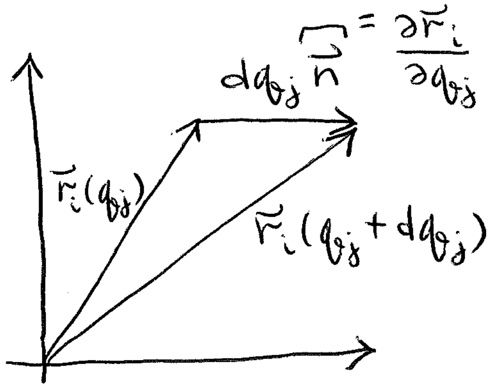
$\boxed{p_j = \text{const}}$ the generalized momentum is conserved

This is only true if q_j's are independent. For ex., in the rolling hoop problem $\mathcal{L}$ did not depend on θ but the constraint imposed $r\dot{\theta} = \dot{x}$, s.t. $p_\theta = mr^2\dot{\theta}$ is <u>not</u> conserved.

Now, consider a generalized coord q_j s.t. $q_j \to q_j + dq_j$ represents a translation of the system. Clearly, T is not affected by this shift $\Rightarrow \frac{\partial T}{\partial q_j} = 0$. Moreover, we shall assume that the system is conservative. Then

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} = \dot{p}_j = \underbrace{-\frac{\partial V}{\partial q_j}}_{Q_j\text{, generalized force}}$$

Recall that $Q_j = \sum_i \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}$



$$\frac{\partial \vec{r}_i}{\partial q_j} = \lim_{dq_j \to 0} \frac{\vec{r}_i(q_j + dq_j) - \vec{r}_i(q_j)}{dq_j} =$$

$$= \lim_{dq_j \to 0} \frac{dq_j}{dq_j}\vec{n} = \underline{\underline{\vec{n}}}$$

$\vec{n}$ is the dir'n of the translation

Hence $Q_j = \sum_j \vec{F}_j \cdot \vec{n} = \vec{F} \cdot \vec{n}$, projection of the total force onto $\vec{n}$.

On the other hand, with $T = \frac{1}{2} \sum_i m_i \dot{\vec{r}}^{\,2}$

we have:

$$p_j = \frac{\partial T}{\partial \dot{q}_j} = \sum_i m_i \dot{\vec{r}}_i \cdot \underbrace{\frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_j}}_{\frac{\partial \vec{r}_i}{\partial q_j}} = \sum_i m_i \vec{v}_i \cdot \underbrace{\frac{\partial \vec{r}_i}{\partial q_j}}_{\vec{n}} =$$

$= \vec{n} \cdot \left(\sum_i m_i \vec{v}_i \right)$, projection of the total momentum onto $\vec{n}$.

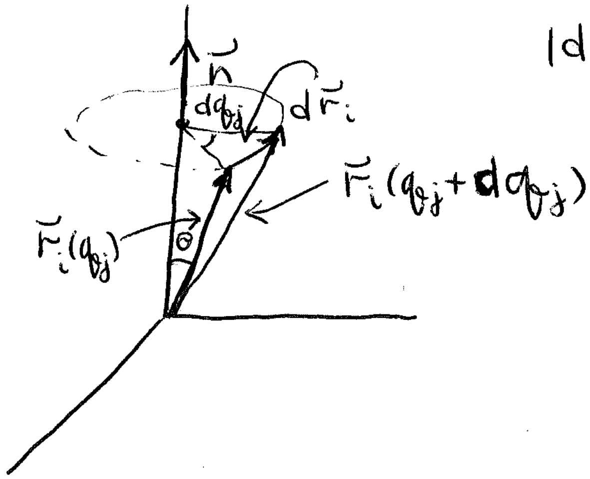
system is inv wrt this translation

If q_j is cyclic, $Q_j = -\frac{\partial V}{\partial q_j} = 0$, and the projection of the total momentum onto $\vec{n}$ is conserved.

If $q_j \to q_j + dq_j$ corresponds to a rotation, $\frac{\partial T}{\partial q_j} = 0$ again, since magnitudes of all velocities are not affected by a rotation.

So we get $\dot{p}_j = Q_j$ again.

Here,



$$|d\vec{r}_i| = r_i \sin\theta \, dq_j$$

$$\Downarrow$$

$$\left|\frac{\partial \vec{r}_i}{\partial q_j}\right| = r_i \sin\theta$$

$$\Downarrow$$

$$\frac{\partial \vec{r}_i}{\partial q_j} = \vec{n} \times \vec{r}_i$$

Then $Q_j = \sum_i \vec{F}_i \cdot (\vec{n} \times \vec{r}_i) = \sum_i \vec{n} \cdot (\vec{r}_i \times \vec{F}_i) =$

$$= \vec{n} \cdot \underbrace{\left(\sum_i \vec{r}_i \times \vec{F}_i\right)}_{\vec{N}\text{, total torque}} = \vec{n} \cdot \vec{N}.$$

Moreover,

$$p_j = \frac{\partial T}{\partial \dot{q}_j} = \sum_i m_i \vec{v}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = \sum_i m_i \vec{v}_i \cdot (\vec{n} \times \vec{r}_i) =$$

$$= \vec{n} \cdot \underbrace{\left(\sum_i \vec{r}_i \times m_i \vec{v}_i\right)}_{\vec{L}\text{, total angular momentum}} = \vec{n} \cdot \vec{L}.$$

system is inv wrt this rotation

If $-\frac{\partial V}{\partial q_j} = Q_j = 0$, ($q_j$ is cyclic) the projection of $\vec{L}$ onto $\vec{n}$ is conserved.

Conservation of energy

Consider $\mathcal{L} = \mathcal{L}(\{q\}, \{\dot{q}\}, t)$:

$$\frac{d\mathcal{L}}{dt} = \sum_j \Big(\underbrace{\frac{\partial \mathcal{L}}{\partial q_j}}_{= \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j}\right) \text{ from LE's}} \dot{q}_j + \frac{\partial \mathcal{L}}{\partial \dot{q}_j} \ddot{q}_j \Big) + \frac{\partial \mathcal{L}}{\partial t} =$$

$$= \sum_j \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j} \dot{q}_j \right) + \frac{\partial \mathcal{L}}{\partial t} \quad , \text{ or}$$

$$\frac{d}{dt}\Big(\underbrace{\sum_j \dot{q}_j \frac{\partial \mathcal{L}}{\partial \dot{q}_j} - \mathcal{L}}_{\equiv h(\{q\}, \{\dot{q}\}, t) \;=\; \text{energy f'n}} \Big) + \frac{\partial \mathcal{L}}{\partial t} = 0 .$$

Now, $\underline{\underline{\dfrac{dh}{dt} = - \dfrac{\partial \mathcal{L}}{\partial t}}}$.

<u>If</u> $\dfrac{\partial \mathcal{L}}{\partial t} = 0 \Rightarrow h$ is conserved

For a wide range of systems,

$$\mathcal{L}(\{q\},\{\dot{q}\},t) = \mathcal{L}_0(\{q\},t) + \mathcal{L}_1(\{q\},\{\dot{q}\},t) + \mathcal{L}_2(\{q\},\{\dot{q}\},t) \quad , \text{ where}$$

$\mathcal{L}_0$ does not depend on $\dot{q}_j$'s while $\mathcal{L}_1$ & $\mathcal{L}_2$ are homogeneous f's of $\dot{q}_j$'s of the 1st & 2nd degree, respectively.

Recall that

$f(\alpha x, \alpha y) = \alpha^k f(x,y)$ is a homog. f'n of degree k, and for such a f'n,

$$\sum_i x_i \frac{\partial f}{\partial x_i} = kf$$

In our case,

$$h = \underbrace{\sum_j \dot{q}_j \frac{\partial \mathcal{L}_1}{\partial \dot{q}_j}}_{\mathcal{L}_1} + \underbrace{\sum_j \dot{q}_j \frac{\partial \mathcal{L}_2}{\partial \dot{q}_j}}_{2\mathcal{L}_2} - \mathcal{L}_0 - \mathcal{L}_1 - \mathcal{L}_2 = \mathcal{L}_2 - \mathcal{L}_0 .$$

Recall that $T = T_2$ if the transform from $\vec{r}_i$ to q_j does not involve t explicitly. Further, if V does not depend on $\dot{q}_j$'s, $V = V_0 \Rightarrow \begin{cases} \mathcal{L}_2 = T, \\ \mathcal{L}_0 = -V \end{cases}$

Then $h = T + V = \underbrace{E}_{\text{total energy}}$

In such systems, $E = \text{const}$ if V does not depend on t explicitly.

__Interlude__: systems with friction

Consider EoM $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j}\right) - \frac{\partial \mathcal{L}}{\partial q_j} = \underbrace{Q_j}$

generalized forces <u>not</u> arising from any potential

For ex., forces of friction:

$$F_x = -k_x v_x \quad , \text{ etc.}$$

Define $\mathcal{F} = \frac{1}{2} \sum_i \left(k_x v_{i,x}^2 + k_y v_{i,y}^2 + k_z v_{i,z}^2 \right)$

sum over particles

Rayleigh's dissipation function

Then $F_{j,x} = -\frac{\partial \mathcal{F}}{\partial v_{j,x}}$, etc. $\forall j$

$$\Downarrow$$

$$\vec{F}_j = -\vec{\nabla}_{\vec{v}_j} \mathcal{F}$$

Work done by the system against friction is, for a single particle, given by

$$dW = -\vec{F}\cdot d\vec{r} = -\vec{F}\cdot\vec{v}\,dt = +\underbrace{(k_x v_x^2 + k_x v_y^2 + k_z v_z^2)}_{=\,2\mathcal{F}}\,dt,$$

$2\mathcal{F}$ = rate of dissipation due to friction

$$\frac{dW}{dt} = 2\mathcal{F}$$

Finally,

$$Q_j = \sum_i \vec{F}_i \cdot \underbrace{\frac{\partial \vec{r}_i}{\partial q_j}}_{\frac{\partial \vec{v}_i}{\partial \dot{q}_j}} = -\sum_i \vec{\nabla}_{\vec{v}_i} \mathcal{F} \cdot \frac{\partial \vec{v}_i}{\partial \dot{q}_j} = -\frac{\partial \mathcal{F}}{\partial \dot{q}_j} .$$

Thus the EoM become:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j}\right) - \frac{\partial \mathcal{L}}{\partial q_j} + \frac{\partial \mathcal{F}}{\partial \dot{q}_j} = 0$$

Consequently,

$$\frac{d\mathcal{L}}{dt} = \frac{\partial \mathcal{L}}{\partial t} + \sum_j \left(\overbrace{\left[\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j}\right) + \frac{\partial \mathcal{F}}{\partial \dot{q}_j} \right]}^{\frac{\partial \mathcal{L}}{\partial q_j}} \dot{q}_j + \frac{\partial \mathcal{L}}{\partial \dot{q}_j} \ddot{q}_j \right) =$$

$$= \sum_j \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j} \dot{q}_j\right) + \sum_j \frac{\partial \mathcal{F}}{\partial \dot{q}_j} \dot{q}_j + \frac{\partial \mathcal{L}}{\partial t} ,$$

$$\frac{dh}{dt} + \frac{\partial \mathcal{L}}{\partial t} + \underbrace{\sum_j \frac{\partial \mathcal{F}}{\partial \dot{q}_j} \dot{q}_j}_{2\mathcal{F}} = 0 .$$

$2\mathcal{F}$ since $\mathcal{F}$ is a homog. f'n of $\dot{q}_j$'s with $k=2$

$$\Downarrow$$

$$\frac{dh}{dt} = -\frac{\partial \mathcal{L}}{\partial t} - 2\mathcal{F} .$$

If $\frac{\partial \mathcal{L}}{\partial t} = 0$, $\frac{dh}{dt} = -2\mathcal{F}$ ← rate of dissipation due to friction