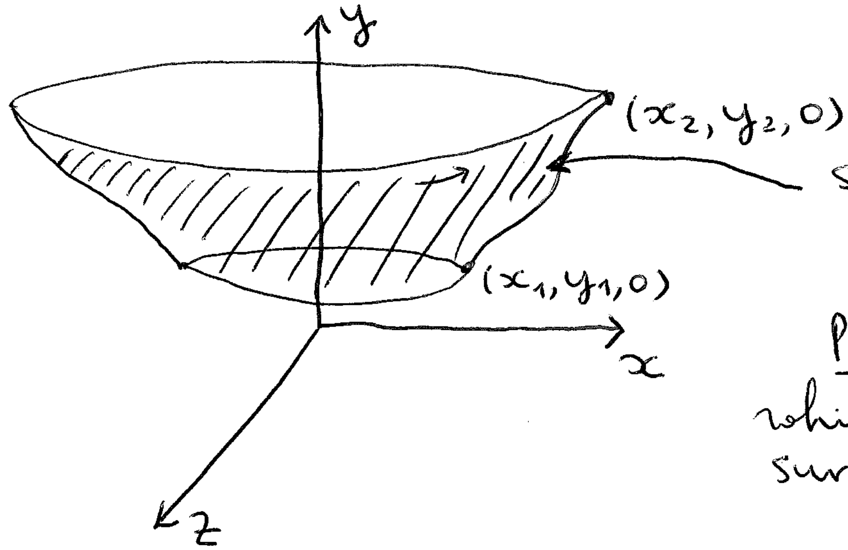
(2.) Minimum surface of revolution



Problem: find $y(x)$ which minimizes this surface

Infinitesimal strip on the surface is given by $2\pi x \underbrace{ds}_{\sqrt{dx^2+dy^2}} = 2\pi x\sqrt{1+\dot{y}^2}\,dx$, and the surface is computed as

$$S = 2\pi \int_1^2 dx\, \underbrace{x\sqrt{1+\dot{y}^2}}_{f = f(\dot{y}, x)}$$

$$\frac{\partial f}{\partial y} = 0, \quad \frac{\partial f}{\partial \dot{y}} = \frac{x\dot{y}}{\sqrt{1+\dot{y}^2}} \quad \text{give}$$

$$\frac{d}{dx}\left(\frac{x\dot{y}}{\sqrt{1+\dot{y}^2}}\right) = 0 \Rightarrow \frac{x\dot{y}}{\sqrt{1+\dot{y}^2}} = a, \text{ or}$$

$$\dot{y}^2(x^2-a^2) = a^2,$$

$$\frac{dy}{dx} = \frac{a}{\sqrt{x^2-a^2}}$$

Finally,

$$y(x) = a\int\frac{dx}{\sqrt{x^2-a^2}} + b = a\cosh^{-1}\frac{x}{a} + b, \text{ or}$$

$$x = a\cosh\frac{y-b}{a}.$$

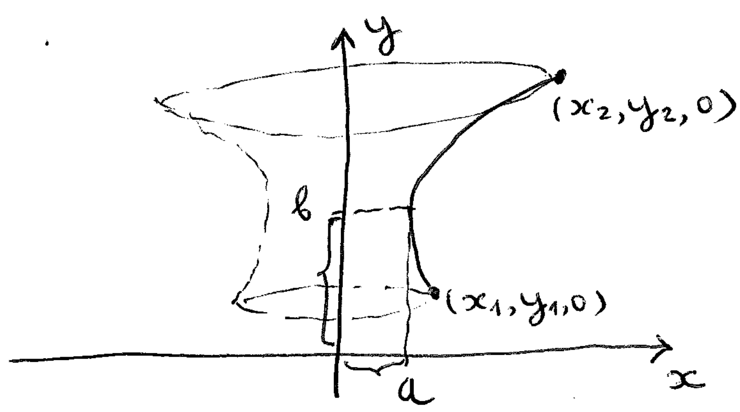
Constants a, b are determined via

$$\begin{cases} x_1 = a\cosh\frac{y_1-b}{a}, \\ x_2 = a\cosh\frac{y_2-b}{a}. \end{cases}$$

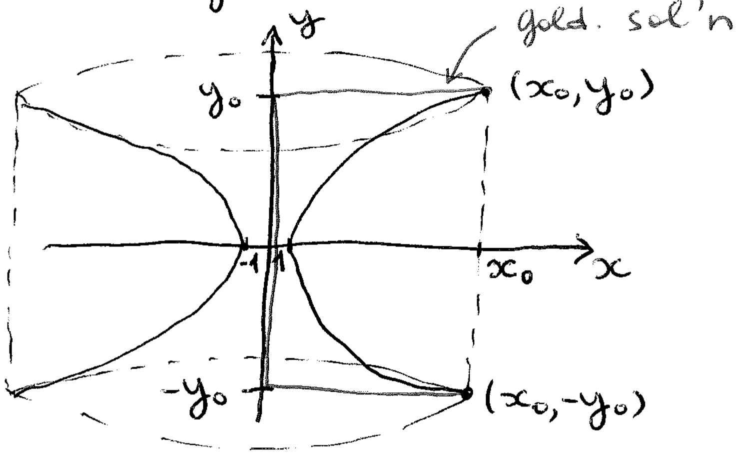
<u>But</u>: sometimes there are no solutions or 2 solutions

Moreover, this is <u>not</u> guaranteed to be a minimum surface (may have found a saddle point). Finally, discontinuous solutions (so-called Goldschmidt solutions) are sometimes better & cannot be found by variational calculus (since the 1st derivatives are discontinuous).

When the solution exists, it looks like this:



Consider a symmetric case:



$$\begin{cases} x_0 = a\cosh \dfrac{y_0 - b}{a}, \\ x_0 = a\cosh \dfrac{-y_0 - b}{a} \end{cases} \Rightarrow \underline{\underline{b=0}}, \text{ and } x_0 = a\cosh\frac{y_0}{a}$$

Now, choose $x_0 = \cosh y_0 \Rightarrow \underline{\underline{a=1}}$

So, $x = \cosh(y)$, leading to

$$S = 2\pi \int_1^2 dx\, x \underbrace{\sqrt{1+\dot{y}^2}}_{= x\dot{y} \text{ since } a=1} = 2\pi \int_1^2 dx\, x^2 \dot{y} \;\text{⊜}$$

$$\text{⊜}\; 2\pi \int_{-y_0}^{y_0} dy \cosh^2(y).$$

If $y_0 = 1$, ($\& \; x_0 = \cosh(1)$) $S = (1 + \sinh(1)\cosh(1)) \times 2\pi \simeq \underline{2.81 \times 2\pi}$.

What about just a straight vertical line from 1 to 2?

$$S_{vert} = 2\pi x_0 \times (\underbrace{2y_0}_{=1}) = 2\cosh(1) \times 2\pi \simeq \underbrace{3.09 \times 2\pi}_{\text{not as good}}.$$

However, the Goldschmidt solution is 2 separate disks:

$$2\times(\pi x_0^2) = \cosh^2(1)\times 2\pi \simeq \underbrace{2.38\times 2\pi}_{\text{significantly better!}}$$

So, the variational solution is not a minimum (certainly not a global minimum since you can "smooth out" the Goldschmidt sol'n to make it continuous)

For many variables, the procedure is the same:

Consider $\delta J = \delta \int_1^2 f(y_1(x), y_2(x), \ldots; \dot{y}_1(x), \dot{y}_2(x), \ldots; x)dx$

and path variations to the extremum problem

$$\begin{cases} y_1(x,\alpha) = \overbrace{y_1(x,0)}^{\text{solutions}} + \alpha\eta_1(x), \\ y_2(x,\alpha) = y_2(x,0) + \alpha\eta_2(x), \\ \ldots \end{cases}$$

$\eta_j(x)$ are continuous through the 2nd derivative and satisfy $\eta_j(x_1) = \eta_j(x_2) = 0, \forall j$.

Then $J = J(\alpha)$ and, consequently,

$$\delta J = \frac{\partial J}{\partial \alpha}d\alpha = \int_1^2 \left(\sum_i \left[\frac{\partial f}{\partial y_i}\frac{\partial y_i}{\partial \alpha} + \frac{\partial f}{\partial \dot{y}_i}\frac{\partial \dot{y}_i}{\partial \alpha}\right]\right) d\alpha\, dx \ominus$$

$$\int_1^2 dx \frac{\partial f}{\partial \dot{y}_i}\frac{\partial^2 y_i}{\partial \alpha \partial x} = \frac{\partial f}{\partial \dot{y}_i}\underbrace{\frac{\partial y_i}{\partial \alpha}}_{\eta_i(x)}\Bigg|_1^2 - \int_1^2 dx \frac{\partial y_i}{\partial \alpha}\frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}_i}\right)$$

$$\ominus \int_1^2 dx \sum_i \left[\frac{\partial f}{\partial y_i} - \frac{\partial}{\partial x} \frac{\partial f}{\partial \dot{y}_i} \right] \frac{\partial y_i}{\partial \alpha} d\alpha .$$

Using $\left(\frac{\partial y_i}{\partial \alpha}\right)_{\alpha=0} d\alpha = \delta y_i$, we obtain:

$$\delta J\Big|_{\alpha=0} = \int_1^2 dx \sum_i \left[\frac{\partial f}{\partial y_i} - \frac{d}{dx} \frac{\partial f}{\partial \dot{y}_i} \right]\Big|_{\alpha=0} \delta y_i .$$

$$\Downarrow$$

$$\boxed{\frac{\partial f}{\partial y_i} - \frac{d}{dx} \frac{\partial f}{\partial \dot{y}_i} = 0}$$ Euler-Lagrange eq's

In mechanics,
$$\begin{cases} x \to t, \\ y_i \to q_i, \\ f(\{y\}, \{\dot{y}\}, x) \to \mathcal{L}(\{q\}, \{\dot{q}\}, t) \end{cases}$$

$$I = \int_1^2 dt\, \mathcal{L}(\{q\}, \{\dot{q}\}, t) \Rightarrow \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = 0$$
$$i = 1, \ldots, n$$

Extensions to systems with explicit constraints

In such systems, not all q_i's are independent. However, we can still use a method of Lagrange multipliers.

For ex., consider

$$f_\alpha(q_1, \ldots, q_n; \dot{q}_1, \ldots, \dot{q}_n) = 0$$
$$\alpha = 1, \ldots, m$$

Then $\sum_{\alpha=1}^{m} \lambda_\alpha f_\alpha = 0$ and we can vary

$$\delta \int_{t_1}^{t_2} dt \left(\mathcal{L} + \sum_{\alpha=1}^{m} \lambda_\alpha f_\alpha \right)$$

and treat q_i's and λ_α's as indep. vars.

In general, $\lambda_\alpha = \lambda_\alpha(q_1, \dots, q_n; \dot{q}_1, \dots, \dot{q}_n; t)$, but we shall assume $\lambda_\alpha = \lambda_\alpha(t)$ for simplicity.

Recall that

$$\delta \int_{t_1}^{t_2} dt\, \mathcal{L} = \int_1^2 dt \sum_k \left(\frac{\partial \mathcal{L}}{\partial q_k} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_k} \right) \delta q_k .$$

Now,

$$\delta \int_{t_1}^{t_2} dt \left(\sum_{\alpha=1}^{m} \lambda_\alpha f_\alpha \right) =$$

$$= \int_{t_1}^{t_2} dt \sum_{\alpha=1}^{m} \lambda_\alpha \sum_k \left[\frac{\partial f_\alpha}{\partial q_k} \frac{\partial q_k}{\partial \alpha} + \frac{\partial f_\alpha}{\partial \dot{q}_k} \frac{\partial \dot{q}_k}{\partial \alpha} \right] d\alpha =$$

note that $\left\{ \frac{\partial \lambda_\alpha}{\partial q_k} = 0, \ \frac{\partial \lambda_\alpha}{\partial \dot{q}_k} = 0 \right.$ by assumption

$$= \int_{t_1}^{t_2} dt \left\{ \sum_\alpha \lambda_\alpha \sum_k \frac{\partial f_\alpha}{\partial q_k} \underbrace{\frac{\partial q_k}{\partial \alpha} d\alpha}_{= \delta q_k} - \sum_{\alpha, k} \frac{d}{dt} \left[\frac{\partial f_\alpha}{\partial \dot{q}_k} \lambda_\alpha \right] \times \underbrace{\frac{\partial q_k}{\partial \alpha} d\alpha}_{\delta q_k} \right\} = \int_{t_1}^{t_2} dt \sum_k \delta q_k \left\{ \sum_\alpha \lambda_\alpha \frac{\partial f_\alpha}{\partial q_k} - \sum_\alpha \left(\frac{\partial f_\alpha}{\partial \dot{q}_k} \frac{d\lambda_\alpha}{dt} + \lambda_\alpha \frac{d}{dt} \left(\frac{\partial f_\alpha}{\partial \dot{q}_k} \right) \right) \right\} \equiv \int_{t_1}^{t_2} dt \sum_k \delta q_k Q_k ,$$

where

$$Q_k = \sum_{\alpha=1}^{m} \left\{ \lambda_\alpha \left[\frac{\partial f_\alpha}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial f_\alpha}{\partial \dot{q}_k} \right) \right] - \frac{d\lambda_\alpha}{dt} \frac{\partial f_\alpha}{\partial \dot{q}_k} \right\}.$$

Finally, $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_k} \right) - \frac{\partial \mathcal{L}}{\partial q_k} = Q_k$

Q_k's are generalized forces of constraint

For example, consider

(1.) $\mathcal{L} = \frac{m}{2}(\dot{x}^2+\dot{y}^2+\dot{z}^2) - V(x,y,z)$,

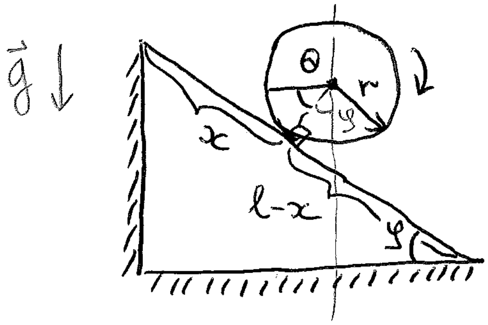
$f(\dot{x},\dot{y},y) = \dot{x}\dot{y} + ky = 0$ ⇐ constraint (k ↑ some const)

Then

$$\begin{cases} m\ddot{x} + \frac{\partial V}{\partial x} = -\lambda\ddot{y} - \dot{\lambda}\dot{y}, \\ m\ddot{y} + \frac{\partial V}{\partial x} = k\lambda - \lambda\ddot{x} - \dot{\lambda}\dot{x}, \\ m\ddot{z} + \frac{\partial V}{\partial z} = 0, \\ \dot{x}\dot{y} + ky = 0. \end{cases}$$

⇒ 4 eq's for $\{x, y, z, \lambda\}$

(2.) a hoop (or a disk) rolling downhill without slipping $\{x, \theta\}$ ⇐ generalized coords



$$\begin{cases} T = \frac{M}{2}\dot{x}^2 + \frac{M}{2}r^2\dot{\theta}^2, \\ V = Mg(l-x)\sin\varphi \end{cases}$$

Constraint: $r\,d\theta = dx$ [no slipping]

Note that the constraint is not given as $f(\theta, x) = 0$, but rather as

$$\underbrace{\frac{\partial f}{\partial \theta}}_{r} d\theta + \underbrace{\frac{\partial f}{\partial x}}_{-1} dx = 0 \qquad \left[\frac{\partial f}{\partial \dot\theta} = 0, \; \frac{\partial f}{\partial \dot x} = 0 \right]$$

Then $\begin{cases} Q_\theta = \lambda r, \\ Q_x = -\lambda \end{cases}$, yielding

$$\left[\mathcal{L} = \frac{M\dot x^2}{2} + \frac{M r^2 \dot\theta^2}{2} - \underbrace{Mg(l-x)\sin\varphi}_{V} \right]$$

$$\begin{cases} (1) \quad M\ddot x - Mg\sin\varphi = -\lambda, \\ (2) \quad Mr^2\ddot\theta = \lambda r \end{cases} + \underbrace{r\dot\theta = \dot x}_{\text{constraint}}$$

$\Downarrow$

find $\{x, \theta, \lambda\}$

Further, $r\ddot\theta = \ddot x \implies \begin{cases} M\ddot x = \lambda, \leftarrow (2) \\ Mg\sin\varphi = 2\lambda \leftarrow (1) \end{cases}$

$$\lambda = \frac{Mg\sin\varphi}{2} \implies \ddot x = \frac{g\sin\varphi}{2}.$$

Finally, $\ddot\theta = \frac{\ddot x}{r} = \frac{g\sin\varphi}{2r}$.

$$|Q_x| = \lambda = \frac{Mg\sin\varphi}{2} \Leftarrow \text{friction force caused by the constraint}$$

Note that for a mass slipping w/out rotation down a frictionless plane,

$$\ddot{x} = g \sin\varphi .$$

Here, $\ddot{x} = \frac{g\sin\varphi}{2}$ b/c ~~[illegible]~~ the rest of the work done by the gravitational field goes into rotating the disk faster & faster.