Applications of the Lagrange approach   Lecture 3

General procedure: write $\mathcal{L} = T - V$ in generalized coords, substitute $\mathcal{L}$ into Lagrange eq's to obtain EoMs.

Thus $T = \sum_i \frac{m_i}{2} \Big(\underbrace{\frac{d\vec{r}_i}{dt}}_{\frac{\partial \vec{r}_i}{\partial t} + \sum_j \frac{\partial \vec{r}_i}{\partial q_j} \dot{q}_j}\Big)^2 =$

$$= \underbrace{\sum_i \frac{m_i}{2}\left(\frac{\partial \vec{r}_i}{\partial t}\right)^2}_{M_0} + \sum_j \dot{q}_j \Big(\underbrace{\sum_i m_i \frac{\partial \vec{r}_i}{\partial t} \cdot \frac{\partial \vec{r}_i}{\partial q_j}}_{M_j}\Big) +$$

$$+ \frac{1}{2} \sum_{j,k} \dot{q}_j \dot{q}_k \Big(\underbrace{\sum_i m_i \frac{\partial \vec{r}_i}{\partial q_j} \cdot \frac{\partial \vec{r}_i}{\partial q_k}}_{M_{jk}}\Big) =$$

If $\vec{r}_i = \vec{r}_i(q_1, \ldots, q_n) \Rightarrow \frac{\partial \vec{r}_i}{\partial t} = 0, \ \forall i$

no explicit t-dependence

$$= \frac{1}{2} \sum_{j,k} M_{jk} \dot{q}_j \dot{q}_k .$$

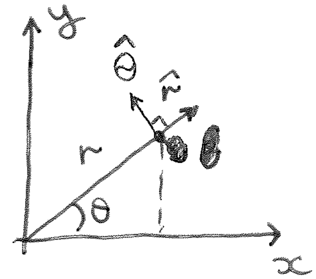
(1.) Single particle: Cartesian coords.

$$T = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \Rightarrow \begin{cases} \frac{\partial T}{\partial \dot{x}} = m\dot{x}, \\ \frac{\partial T}{\partial x} = 0 \end{cases} \quad \text{etc.}$$

$$\underbrace{-\frac{\partial V}{\partial x}}_{F_x} - m\ddot{x} = 0 \Rightarrow m\ddot{x} = F_x \quad \text{same for } y, z$$

② Single particle: polar coords.

$\boxed{2D}$ $\begin{cases} x = r\cos\theta, \\ y = r\sin\theta \end{cases}$



Then $\begin{cases} \dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta, \\ \dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta \end{cases}$

$$T = \frac{m}{2}(\dot{x}^2+\dot{y}^2) = \frac{m}{2}\Big[\underline{\dot{r}^2\cos^2\theta} - 2r\dot{r}\dot{\theta}\cancel{\sin\theta}\cos\theta + r^2\dot{\theta}^2\sin^2\theta + \underline{\dot{r}^2\sin^2\theta} + 2r\dot{r}\dot{\theta}\cancel{\sin\theta}\cos\theta + r^2\dot{\theta}^2\cos^2\theta\Big] =$$

$$= \underbrace{\frac{m}{2}\left[\dot{r}^2 + r^2\dot{\theta}^2\right]}_{\text{quadratic in } \dot{r},\ \dot{\theta}}.$$

Note that $d\vec{r} \overset{\hat{x}dx + \hat{y}dy}{=} \underbrace{\hat{r}}_{\frac{\partial \vec{r}}{\partial r}}dr + \underbrace{r\hat{\theta}}_{\frac{\partial \vec{r}}{\partial \theta}}d\theta \Rightarrow \vec{v} = \frac{d\vec{r}}{dt} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$

Recall that $Q_j = \underbrace{\sum_i}_{\text{sum over particles}} \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}$ and

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j.$$

Here, $\begin{cases} Q_r = \vec{F}\cdot\frac{\partial \vec{r}}{\partial r} = \vec{F}\cdot\hat{r} = F_r, \\ Q_\theta = \vec{F}\cdot\frac{\partial \vec{r}}{\partial \theta} = \vec{F}\cdot(r\hat{\theta}) = rF_\theta. \end{cases}$

Then $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot r}\right) - \frac{\partial T}{\partial r} = Q_r$, or

$$m\ddot r - m r \dot\theta^2 = F_r \Rightarrow m\ddot r = F_r + \underbrace{m r \dot\theta}_{\text{centripetal force}}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot\theta}\right) - \frac{\partial T}{\partial \theta} = Q_\theta, \text{ or}$$

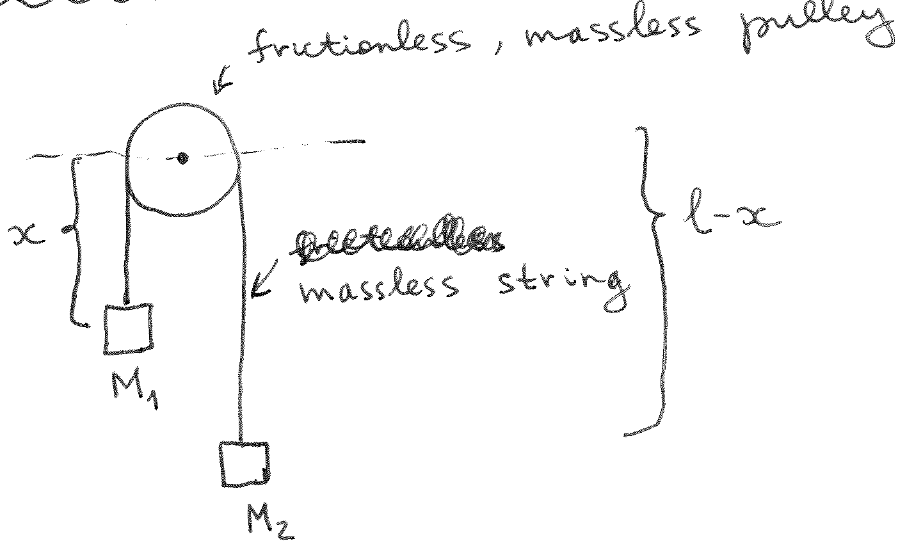
$$\frac{d}{dt}\left(m r^2 \dot\theta\right) = r F_\theta,$$

$$m r^2 \ddot\theta + 2 m r \dot r \dot\theta = \underbrace{r F_\theta}_{\text{torque } N}.$$

$$L = |\vec r \times \vec p_\theta| = m\,|\underbrace{\vec r}_{r\hat r} \times \underbrace{\vec v_\theta}_{r\dot\theta\hat\theta}| = m r^2 \dot\theta$$

So, we obtain $\underbrace{\frac{dL}{dt} = N}_{\text{torque eq'n}}$

③ Atwood's machine



x is the only indep. coord due to a holonomic constraint: $x_1 + x_2 = l \Rightarrow \begin{cases} x_1 = x, \\ x_2 = l - x. \end{cases}$

$$\begin{cases} V = -M_1 g x - M_2 g(l-x), \\ T = \dfrac{M_1 \dot{x}_1^2}{2} + \dfrac{M_2 \dot{x}_2^2}{2} = \dfrac{M_1+M_2}{2}\dot{x}^2 \end{cases}$$

$$\begin{cases} \dfrac{\partial L}{\partial x} = -\dfrac{\partial V}{\partial x} = (M_1 - M_2) g, \\ \dfrac{\partial L}{\partial \dot{x}} = (M_1+M_2)\dot{x} \end{cases} \Rightarrow \underbrace{(M_1+M_2)\ddot{x} = (M_1-M_2)g}_{\text{EoM}}$$

$\ddot{x} = 0$ if $\underbrace{M_1 = M_2}_{\text{total force } = 0}$

$\Uparrow$

note that constraint forces (i.e., tension of the rope) do not appear explicitly goldstein Ch. 2

Hamilton's principle

We've derived Lagrange's equations (LEs) using
(i) instantaneous state of the system and
(ii) the concept of small virtual displacements.

It's also possible to obtain LEs by considering the actual motion of the system between t_1 & t_2 and small virtual variations around the actual path.

Configuration space: $\underbrace{q_1, \dots, q_n}_{\text{generalized coords}}$

With time, the system moves in the configuration space, tracing out a path.

For conservative systems, the motion from t_1 to t_2 is such that

$$I = \int_{t_1}^{t_2} \mathcal{L}\, dt \quad \text{is stationary for the actual path}$$

$\Rightarrow$ Hamilton's principle

$\mathcal{L} = T - V$ is the Lagrangian

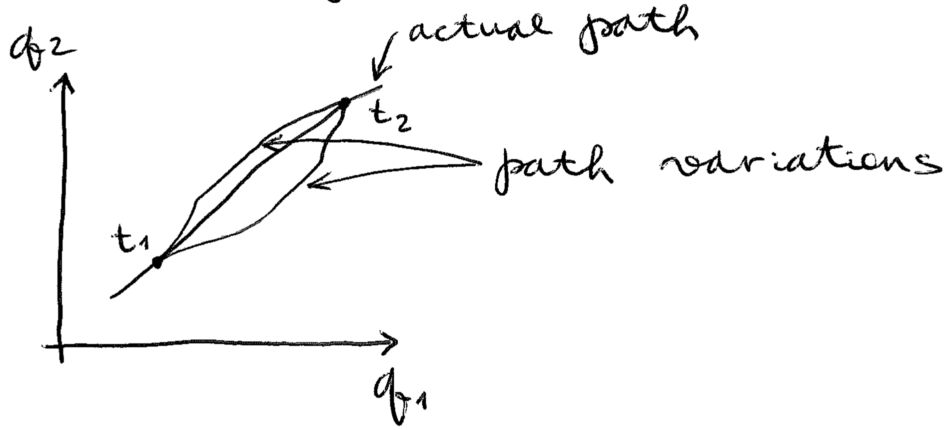
In other words,

$$(*) \qquad \delta I = \delta \int_{t_1}^{t_2} \mathcal{L}(q_1, \dots, q_n; \dot{q}_1, \dots, \dot{q}_n; t)\, dt = 0$$

LEs follow from $(*)$

Hamilton's principle remains true even if $V = V(\{q\}, \{\dot{q}\}, t)$ instead of $V = V(\{q\})$ in conservative systems

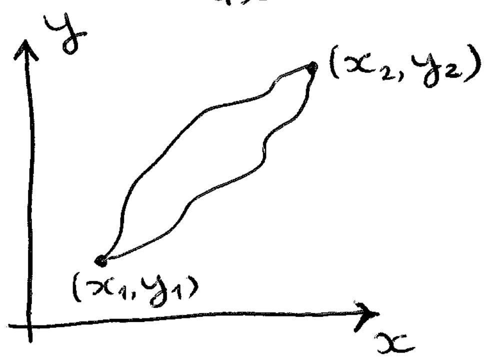
↑ generalized scalar potential



Calculus of variations

Consider $f(y, \dot{y}, x)$ defined on a 1D curve $y(x)$:

$\dot{y} \equiv \frac{dy}{dx}$



Note that configuration space is 1D here: y is the only generalized coord, and x is like t

We wish to find $y(x)$ s.t. the 1D integral $J = \int_{x_1}^{x_2} dx\, f(y, \dot{y}, x)$ is stationary

We choose to parameterize paths as follows:

$$y(x, \alpha) = \underbrace{y(x, 0)}_{\text{actual path}} + \alpha \underbrace{\eta(x)}_{\text{arbitrary f'n (well-behaved)}}$$

Note that $\eta(x_1) = \eta(x_2) = 0$

Now, $$J = J(\alpha) = \int_{x_1}^{x_2} dx \; f\left(y(x,\alpha), \dot{y}(x,\alpha), x\right),$$

and the stationary condition is

$$\left.\frac{dJ}{d\alpha}\right|_{\alpha=0} = 0 \qquad (**)$$

$$\frac{dJ}{d\alpha} = \int_{x_1}^{x_2} dx \left[\frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial \dot{y}}\frac{\partial \dot{y}}{\partial \alpha}\right] =$$

$$= \int_{x_1}^{x_2} dx \left[\frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha} + \underbrace{\frac{\partial f}{\partial \dot{y}}\frac{\partial^2 y}{\partial x\, \partial \alpha}}_{\text{by parts}}\right] =$$

$$= \int_{x_1}^{x_2} dx \left[\frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha}\right] + \left.\frac{\partial f}{\partial \dot{y}}\cancel{\frac{\partial y}{\partial \alpha}}\right|_{x_1}^{x_2} - \int_{x_1}^{x_2} dx \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right)\frac{\partial y}{\partial \alpha} =$$

0 since

$$\frac{\partial y}{\partial \alpha} = \eta(x) \quad \& \quad \eta(x_1) = \eta(x_2) = 0$$

$$= \int_{x_1}^{x_2} dx \left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial \dot{y}}\right)\frac{\partial y}{\partial \alpha}.$$

Condition $(**)$ then gives

$$\int_{x_1}^{x_2} dx \left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial \dot{y}}\right)\left.\underbrace{\frac{\partial y}{\partial \alpha}}_{\eta(x)}\right|_{\alpha=0} = 0$$

$$\Downarrow$$

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial \dot{y}} = 0. \qquad (***)$$

If we define $\left(\frac{\partial y}{\partial \alpha}\right)_{\alpha=0} d\alpha = \eta(x) d\alpha \equiv \delta y$ and

$\left(\frac{dJ}{d\alpha}\right)_{\alpha=0} d\alpha \equiv \delta J$, we obtain:

$$\delta J = \int_{x_1}^{x_2} dx \left(\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial \dot{y}} \right) \delta y = 0 ,$$

which again leads to (***).

Ex.: (1.) Two points on a plane:

$ds = \sqrt{dx^2 + dy^2}$, and the total length is given by

$$I = \int_1^2 ds = \int_{x_1}^{x_2} dx \underbrace{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}_{f(y, \dot{y}, x)\ [= f(\dot{y}) \text{ here}]}$$

$$\frac{\partial f}{\partial y} = 0 , \quad \frac{\partial f}{\partial \dot{y}} = \frac{\dot{y}}{\sqrt{1+\dot{y}^2}} \quad \text{give}$$

$$\frac{d}{dx}\left(\frac{\dot{y}}{\sqrt{1+\dot{y}^2}} \right) = 0 \Rightarrow \frac{\dot{y}}{\sqrt{1+\dot{y}^2}} = C_1 , \text{ or}$$

$$\dot{y}^2 = C_1^2 (1 + \dot{y}^2) ,$$

$$\dot{y} = \frac{C_1}{\sqrt{1 - C_1^2}} \equiv a .$$

So, $y(x) = ax + b \Leftarrow$ straight line, as expected