Lecture 2

Constraints

Ex.: 1) rigid bodies (all r_{ij} distances unchanged)
2) particle placed on the surface of a solid sphere

Holonomic constraints:

$$f(\vec{r}_1, \dots, \vec{r}_N, t) = 0 \qquad (*)$$

→ Rigid body: $(\vec{r}_i - \vec{r}_j)^2 = C_{ij}^2 \quad , \forall i,j \text{ s.t. } j \neq i$

→ a particle constrained to move along any curve or on any surface

However, a particle in Ex. (2) is described by a non-holonomic constraint:

$$\underbrace{r^2 \geq a^2}_{}$$
sphere radius

[Constraints of type (*) (i.e., holonomic) may or may not be explicitly time-dependent.]

With constraints, not all $\vec{r}_i$'s are independent. Also, the forces of constraint are not explicitly known.

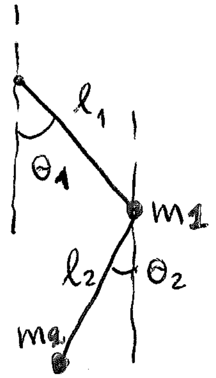
The first problem is solved by introducing

$3N - k$ generalized coords $\Rightarrow q_1, q_2, \dots, q_{3N-k}$:

N: # particles; k: # holonomic constraints

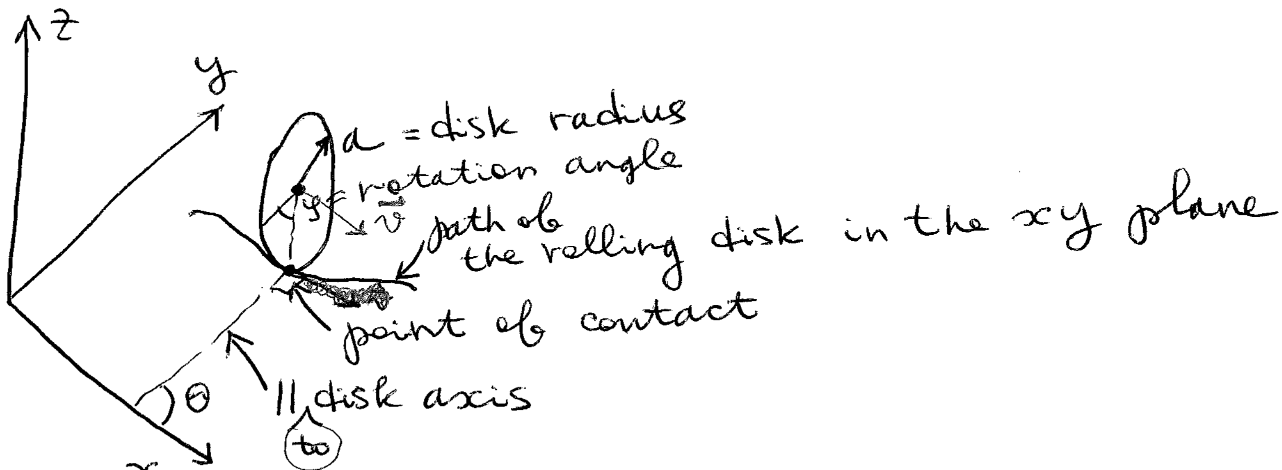
transform'n eq's $\Rightarrow$
$$\begin{cases} \vec{r}_1 = \vec{r}_1(q_1, \dots, q_{3N-k}, t) \\ \vdots \\ \vec{r}_N = \vec{r}_N(q_1, \dots, q_{3N-k}, t) \end{cases}$$

Generalized coords are problem-dependent:
e.g. for a double pendulum



$\{\theta_1, \theta_2\}$ is an obvious choice

Consider now a disk rolling on its edge on a flat plane:



$$v = a\dot{\varphi} \qquad \begin{cases} \dot{x} = v\sin\theta, \\ \dot{y} = -v\cos\theta \end{cases}$$

Then
$$\begin{cases} dx - a\sin\theta \, d\varphi = 0, \\ dy + a\cos\theta \, d\varphi = 0 \end{cases} \qquad (**)$$

(**) cannot be integrated in general, so cannot yield smth. like $f(x, y, \theta, \varphi) = 0$.
Thus, (**) are non-holonomic constraints.

Specifically, if

$$\sum_{i=1}^{n} g_i(x_1, \ldots, x_n)\, dx_i = 0$$ corresponds to

$df = 0$ for some $f = f(x_1, \ldots, x_n)$, we must have $g_i = \dfrac{\partial f}{\partial x_i}$.

But then $\dfrac{\partial g_i}{\partial x_j} = \dfrac{\partial^2 f}{\partial x_i \partial x_j} = \underline{\underline{\dfrac{\partial g_j}{\partial x_i}}}$ $\forall i, j$.

In our case, (**) yield:

$$dx - a \sin\theta \, d\varphi + \emptyset \cdot d\theta + \emptyset \cdot dy = 0$$

$$\frac{\partial}{\partial \theta}\left(\frac{\partial f}{\partial \varphi}\right) = \frac{\partial}{\partial \theta}(-a\sin\theta) = a\cos\theta$$

$$\not\parallel$$

$$\frac{\partial}{\partial \varphi}\left(\frac{\partial f}{\partial \theta}\right) = \frac{\partial}{\partial \varphi}(\emptyset) = 0 .$$

So, the 1st eq'n in (**) corresponds to a non-holonomic constraint.

The 2nd eq'n in (**) is

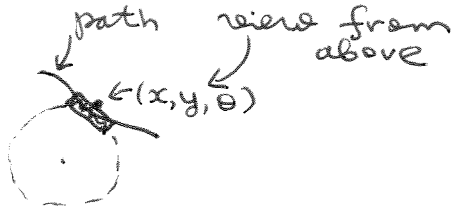
$$\emptyset \cdot dx + dy + a\cos\theta \, d\varphi + \emptyset \cdot d\theta = 0$$

Again, $\dfrac{\partial}{\partial \varphi}\left(\dfrac{\partial f}{\partial \theta}\right) = 0$ while

$$\frac{\partial}{\partial \theta}\left(\frac{\partial f}{\partial \varphi}\right) = \frac{\partial}{\partial \theta}(a\cos\theta) = -a\sin\theta \neq 0 \text{ in general}$$

So this constraint is also non-holonomic.

Physically, starting from any (x, y, θ), the disk can be rolled in a circle, of an arbitrary radius:
tangent to the path



This will increment φ by an arbitrary amount while leaving (x, y, θ) the same. Thus there's no function

$f(x, y, \theta, \varphi) = \text{const}$, unless it does not depend on φ.

D'Alembert's principle and Lagrange's equations

Consider a mechanical system at time t, subject to forces & constraints. A **virtual displacement** $\delta\vec{r}_i$ of particle i is a change in the particle's position in time δt *if* the forces & constraints are "frozen" to their values @ time t.

If the system is at *equilibrium*,

$$\underbrace{\vec{F}_i}_{\substack{\text{total force}\\ \text{on particle } i}} = 0 \,, \quad \forall i \quad \Rightarrow \quad \sum_i \vec{F}_i \cdot \delta\vec{r}_i = 0$$

Since $\vec{F}_i = \vec{F}_i^{(a)} + \vec{f}_i$, we have:

$$\sum_i \vec{F}_i^{(a)} \cdot \delta\vec{r}_i + \sum_i \vec{f}_i \cdot \delta\vec{r}_i = 0 \,.$$

Now, assume that $\sum_i \vec{f}_i \cdot \delta\vec{r}_i = 0$
[true for rigid bodies or for particles constrained to move along curves or on surfaces, since in the latter case $\vec{f}_i \perp \delta\vec{r}_i$, $\forall i$. The constraints may even be t-dependent since in the concept of virtual displacements all constraints are "frozen" @ t.]

In this case,

$$\sum_i \vec{F}_i^{(a)} \cdot \delta\vec{r}_i = 0$$ principle of virtual work, system @ equil.

However, unlike $\vec{F}_i$, $\vec{F}_i^{(a)} \neq 0$ in general; not all $\delta\vec{r}_i$'s are independent since they are connected by the constraints.

When the system is moving,

$\vec{F}_i = \dot{\vec{p}}_i$ gives $\sum_i (\vec{F}_i - \dot{\vec{p}}_i)\cdot\delta\vec{r}_i = 0$, or

$$\sum_i (\vec{F}_i^{(a)} - \dot{\vec{p}}_i)\cdot\delta\vec{r}_i + \underbrace{\sum_i \vec{f}_i \cdot \delta\vec{r}_i}_{\text{assume this is } =0} = 0$$

$$\Downarrow$$

$$\sum_i (\vec{F}_i^{(a)} - \dot{\vec{p}}_i)\cdot\delta\vec{r}_i = 0.$$ D'alembert's principle

Recall that

$$\vec{r}_i = \vec{r}_i(q_1, \ldots, \underbrace{q_n}_{\#\text{ indep. coords}}, t).$$

Then $$\vec{v}_i = \frac{d\vec{r}_i}{dt} = \frac{\partial\vec{r}_i}{\partial t} + \sum_k \frac{\partial\vec{r}_i}{\partial q_k}\dot{q}_k$$

Likewise, $$\delta\vec{r}_i = \sum_j \frac{\partial\vec{r}_i}{\partial q_j}\delta q_j$$

no δt by definition

Consequently,

$$\sum_i \vec{F}_i^{(a)} \cdot \delta \vec{r}_i = \sum_{i,j} \vec{F}_i^{(a)} \cdot \frac{\partial \vec{r}_i}{\partial q_j} \delta q_j = \sum_j \underbrace{Q_j}_{\parallel} \delta q_j$$

generalized force $\displaystyle \sum_i \vec{F}_i^{(a)} \cdot \frac{\partial \vec{r}_i}{\partial q_j}$

Further, $\displaystyle \sum_i \dot{\vec{p}}_i \cdot \delta \vec{r}_i = \sum_i m_i \ddot{\vec{r}}_i \cdot \delta \vec{r}_i =$

$$= \sum_{i,j} m_i \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \delta q_j .$$

Now, consider

$$\sum_i m_i \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = \sum_i \left[\frac{d}{dt} \left(m_i \dot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \right) - m_i \dot{\vec{r}}_i \cdot \frac{d}{dt} \left(\frac{\partial \vec{r}_i}{\partial q_j} \right) \right] \boxed{\equiv}$$

$$\frac{d}{dt} \left(\frac{\partial \vec{r}_i}{\partial q_j} \right) = \frac{\partial^2 \vec{r}_i}{\partial q_j \partial t} + \sum_k \frac{\partial^2 \vec{r}_i}{\partial q_j \partial q_k} \dot{q}_k = \frac{\partial \vec{v}_i}{\partial q_j} .$$

Moreover,

$$\frac{\partial \vec{v}_i}{\partial \dot{q}_j} = \frac{\partial}{\partial \dot{q}_j} \left[\sum_k \frac{\partial \vec{r}_i}{\partial q_k} \dot{q}_k + \frac{\partial \vec{r}_i}{\partial t} \right] = \underline{\underline{\frac{\partial \vec{r}_i}{\partial q_j}}} .$$

Thus we have

$$\boxed{\equiv} \sum_i \left[\frac{d}{dt} \left(m_i \vec{v}_i \cdot \frac{\partial \vec{v}_i}{\partial \dot{q}_j} \right) - m_i \vec{v}_i \cdot \frac{\partial \vec{v}_i}{\partial q_j} \right] .$$

Finally,

$$\sum_i (\vec{F}_i^{(a)} - \dot{\vec{p}}_i) \cdot \delta \vec{r}_i = \sum_j Q_j \, \delta q_j =$$

$$= \sum_j \delta q_j \left\{ \sum_i \left[\frac{d}{dt} \left(\frac{m_i}{2} \frac{\partial v_i^2}{\partial \dot{q}_j} \right) - \frac{m_i}{2} \frac{\partial v_i^2}{\partial q_j} \right] \right\} = 0, \text{ or}$$

$$\sum_j \delta q_j \left\{ \frac{d}{dt} \left[\frac{\partial}{\partial \dot{q}_j} \left(\underbrace{\sum_i \frac{m_i}{2} v_i^2}_{T\text{, total kinetic energy of the system}} \right) \right] - \frac{\partial}{\partial q_j} \left[\sum_i \frac{m_i}{2} v_i^2 \right] - Q_j \right\} = 0 .$$

So, $$\sum_j \delta q_j \left\{ \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_i} - Q_j \right\} = 0 .$$

For indep. virtual displacements δq_j, we obtain:

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} = Q_j , \quad j = 1, .., n$$

For conservative forces,

$$\vec{F}_i^{(a)} = - \vec{\nabla}_i V , \quad \text{and} \quad Q_j = \sum_i \vec{F}_i^{(a)} \cdot \frac{\partial \vec{r}_i}{\partial q_j} =$$

$$= - \sum_i \vec{\nabla}_i V \cdot \frac{\partial \vec{r}_i}{\partial q_j} = - \frac{\partial V}{\partial q_j} , \text{ yielding}$$

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial (T - V)}{\partial q_j} = 0 .$$

Moreover, V does not depend on $\dot{q}_j$:

$$(*) \quad \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j}\right) - \frac{\partial \mathcal{L}}{\partial q_j} = 0 \quad \text{, where}$$

← Lagrange eq's

$\mathcal{L} = T - V$ is the Lagrangian

Note that $\mathcal{L}$ satisfying EoM $(*)$ is <u>not</u> unique: $\mathcal{L}'(\{q\},\{\dot{q}\},t) = \mathcal{L}(\{q\},\{\dot{q}\},t) + \frac{dF(\{q\},t)}{dt}$

will also satisfy Lagrange eq's.

Indeed, $$\begin{cases} \dfrac{\partial \mathcal{L}'}{\partial q_i} = \dfrac{\partial \mathcal{L}}{\partial q_i} + \dfrac{\partial}{\partial q_i}\dfrac{dF}{dt}, \\[2ex] \dfrac{\partial \mathcal{L}'}{\partial \dot{q}_i} = \dfrac{\partial \mathcal{L}}{\partial \dot{q}_i} + \dfrac{\partial}{\partial \dot{q}_i}\dfrac{dF}{dt}. \end{cases}$$

$$\frac{dF}{dt} = \sum_i \frac{\partial F}{\partial q_i}\dot{q}_i + \frac{\partial F}{\partial t} \Rightarrow \frac{\partial}{\partial \dot{q}_j}\frac{dF}{dt} = \underline{\underline{\frac{\partial F}{\partial q_j}}}.$$

Then $$\frac{d}{dt}\frac{\partial \mathcal{L}'}{\partial \dot{q}_j} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_j} + \frac{d}{dt}\underbrace{\frac{\partial}{\partial \dot{q}_j}\frac{dF}{dt}}_{\frac{\partial F}{\partial q_j}}.$$

Finally, $$\frac{\partial \mathcal{L}'}{\partial q_j} - \frac{d}{dt}\frac{\partial \mathcal{L}'}{\partial \dot{q}_j} = \frac{\partial \mathcal{L}}{\partial q_j} + \cancel{\frac{\partial}{\partial q_j}\frac{dF}{dt}} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_j} - \cancel{\frac{d}{dt}\frac{\partial F}{\partial q_j}} = 0.$$

So $\mathcal{L}'$ satisfies the Lagrange eq's <u>as well</u>.