Lecture 23

Equivalence of canonical invariance of Poisson brackets (PB) and of the generating function formalism

Consider $n=1$ for simplicity:

$$\begin{cases} Q = Q(p,q), \\ P = P(p,q) \end{cases} \Rightarrow \begin{cases} p = \varphi(q, Q), \\ P = \Psi(q, Q) \end{cases}$$

Now, $$\frac{\partial Q}{\partial Q} = 1 \underset{Q = Q(q,\, \underbrace{\varphi(q,Q)}_{p})}{\Longrightarrow} \frac{\partial Q}{\partial p}\frac{\partial \varphi}{\partial Q} = 1. \quad (*)$$

Next, $$\underbrace{[Q, P] = 1}_{\text{canonical invariance, follows from the symplectic condition}} = \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial P}{\partial q}\frac{\partial Q}{\partial p} =$$

$$= \frac{\partial Q}{\partial q}\frac{\partial \Psi}{\partial Q}\frac{\partial Q}{\partial p} - \frac{\partial Q}{\partial p}\left(\frac{\partial \Psi}{\partial q} + \frac{\partial \Psi}{\partial Q}\frac{\partial Q}{\partial q}\right) =$$

$$\begin{cases} P = \Psi(q, Q), \\ Q = Q(q, p) \end{cases}$$

$$= \frac{\partial \Psi}{\partial Q}\underbrace{\left[\frac{\partial Q}{\partial q}\frac{\partial Q}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial Q}{\partial q}\right]}_{=0} - \frac{\partial Q}{\partial p}\frac{\partial \Psi}{\partial q},$$ so that

$$-\frac{\partial Q}{\partial p}\frac{\partial \Psi}{\partial q} = 1 = \frac{\partial Q}{\partial p}\frac{\partial \varphi}{\partial Q} \Leftarrow (*)$$

Finally, we get $$\frac{\partial \varphi}{\partial Q} = -\frac{\partial \Psi}{\partial q}.$$

On the other hand, if the generating f'n $F_1 = F_1(q, Q)$ exists, we expect:

$$\begin{cases} p = \dfrac{\partial F_1(q,Q)}{\partial q}, \\ P = -\dfrac{\partial F_1(q,Q)}{\partial Q}. \end{cases} \Rightarrow \left.\frac{\partial p}{\partial Q}\right. = -\frac{\partial P}{\partial q}, \text{ or}$$

$$\frac{\partial \varphi}{\partial Q} = -\frac{\partial \psi}{\partial q}, \text{ same as above}$$

Thus the symplectic condition and the generating function approaches are equivalent.

EoM, ICTs and conservation laws

Consider $u = u(p, q, t)$:

$$\frac{du}{dt} = \frac{\partial u}{\partial q_i}\dot{q}_i + \frac{\partial u}{\partial p_i}\dot{p}_i + \frac{\partial u}{\partial t} =$$

$$= \frac{\partial u}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial H}{\partial q_i} + \frac{\partial u}{\partial t} =$$

$$= [u, H] + \frac{\partial u}{\partial t}. \qquad (**)$$

In matrix form,

$$\frac{du}{dt} = \frac{\partial u}{\partial \vec{\eta}}\dot{\vec{\eta}} + \frac{\partial u}{\partial t} = \overbrace{\underbrace{\frac{\partial u}{\partial \vec{\eta}} J \frac{\partial H}{\partial \vec{\eta}}}_{\frac{\partial u}{\partial \eta_i} J_{ij} \frac{\partial H}{\partial \eta_j}}}^{[u, H]_{\vec{\eta}}} + \frac{\partial u}{\partial t},$$ so that (**) is again recovered

Eq. (**) is a generalized EoM for $u(p,q,t)$.

For example, if $u \to q_i$:

$$\dot{q}_i = [q_i, H].$$

If $u \to p_i$: $\dot{p}_i = [p_i, H]$.

In symplectic notation, $\dot{\vec{\eta}} = [\vec{\eta}, H]$.

Indeed, $[\vec{\eta}, H] = \underbrace{\frac{\partial \vec{\eta}}{\partial \vec{\eta}}}_{=1} J \frac{\partial H}{\partial \vec{\eta}} = J \frac{\partial H}{\partial \vec{\eta}}$, so that

$$\dot{\vec{\eta}} = J \frac{\partial H}{\partial \vec{\eta}} \Leftarrow \text{EoM from before}$$

Finally, if $u \to H$: $\frac{dH}{dt} = \frac{\partial H}{\partial t}$.

Note that Eq. (**) is canonically inv.

If $u = \underset{\text{of motion}}{\text{const}}$, Eq. (**) yields

$$\frac{\partial u}{\partial t} = [H, u].$$

If $\frac{\partial u}{\partial t} = 0 \Rightarrow [H, u] = 0$

If $u = \underset{\text{of motion}}{\text{const}}$, $v = \underset{\text{of motion}}{\text{const}}$ and

$\frac{\partial u}{\partial t} = 0$, $\frac{\partial v}{\partial t} = 0$, we can use the Jacobi identity to obtain new constants of the motion:

$$[u,\underbrace{[v,H]}_{=0}] + [v,\underbrace{[H,u]}_{=0}] + [H,[u,v]] = 0 \text{, or}$$

$[H,[u,v]] = 0 \Rightarrow [u,v]$ is also a const of motion.

This even works if $\frac{\partial u}{\partial t} \neq 0$, $\frac{\partial v}{\partial t} \neq 0$ (Poisson's theorem), and can be used to generate new constants of motion.

The PBs can be used to reformulate ICT formalism:

consider $\vec{\xi} = \vec{\eta} + \underbrace{\delta\vec{\eta}}_{\epsilon J \frac{\partial G(\vec{\eta})}{\partial \vec{\eta}}}$ as before

Now, use $[\vec{\eta}, G] = J\frac{\partial G}{\partial \vec{\eta}}$, so that

$$\delta\vec{\eta} = \epsilon\,[\vec{\eta}, G]$$

Now let $G = H$ and substitute $\epsilon \to dt$:

$$\delta\vec{\eta} = dt\,\underbrace{[\vec{\eta}, H]}_{\text{with } H \text{ as } G} \overset{\text{EoMs for } \dot{\vec{\eta}}}{=} dt\,\frac{d\vec{\eta}}{dt} = d\vec{\eta}$$

So, the ICT transform changes

$$\begin{cases} p_i(t) \to p_i(t+dt) = p_i(t) + dp_i \\ q_i(t) \to q_i(t+dt) = q_i(t) + dq_i \end{cases}$$

Correspondingly, the succession of these ICTs generates the motion of the system.

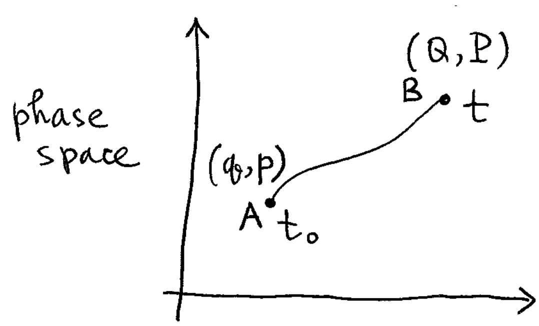
We say that H is the generator of system motion with time: a formal solution to <u>any</u> motion of <u>any</u> mechanical system consists of applying an infinite series of ICTs with H as the generator, starting from some initial conditions $p_i(t_0)$, $q_i(t_0)$, $\forall i$.

Two flavors of canonical transforms:

(1.) <u>passive</u>: $(q,p) \Rightarrow (Q,P)$, like a change of coordinates [usually t-indep.]

(2.) <u>active</u>: $(q,p) \Rightarrow (Q,P)$ expresses a transformation of the <u>same</u> system in time:

$$\begin{cases} \overset{p_i}{\overset{\shortparallel}{p_i(t_0)}}, \\ \underset{q_i}{\underset{\shortparallel}{q_i(t_0)}} \end{cases} \Rightarrow \begin{cases} Q_i = q_i(p(t_0), q(t_0), t), \\ P_i = p_i(p(t_0), q(t_0), t) \end{cases} \qquad t > t_0$$



Define $\underbrace{\partial u}_{\text{change in some f'n } u \text{ under an active transform}} = u(B) - u(A)$ (i.e. active ICT)

Then $\quad \partial u = u(\vec{\eta} + \delta\vec{\eta}) - u(\vec{\eta}) =$

$$= \frac{\partial u}{\partial \vec{\eta}} \underbrace{\delta \vec{\eta}}_{\epsilon J \frac{\partial G}{\partial \vec{\eta}}} = \epsilon \frac{\partial u}{\partial \vec{\eta}} J \frac{\partial G}{\partial \vec{\eta}} = \epsilon [u, G]$$

If $\quad u \to \vec{\eta} : \quad \partial \vec{\eta} = \underbrace{\epsilon [\vec{\eta}, G]}_{\delta \vec{\eta}} = \delta \vec{\eta}$

The infinitesimal change between A & B is just a result of using G as an ICT generator, as expected.

With H, we have to recall that in general $\quad {}_{(1)}\, K = H + \frac{\partial F}{\partial t}$, where

$$F = F_2(q, P, t) = q_i P_i + \epsilon G(q, P, t) \Rightarrow \underbrace{\frac{\partial F}{\partial t} = \epsilon \frac{\partial G}{\partial t}}_{\text{for ICT}}$$

In other words, H is not a "fixed" function but something that needs to be transformed using Eq. (1) to satisfy canonical EoM.

Then we transform $K(A) = H(A) + \epsilon \frac{\partial G}{\partial t}$ into $H(B)$:

$$\partial H = H(B) - K(A) = H(B) - H(A) - \epsilon \frac{\partial G}{\partial t}, \text{ or}$$

$$\partial H = \epsilon \underbrace{[H, G]}_{-[G,H]} - \epsilon \frac{\partial G}{\partial t} = -\epsilon \frac{dG}{dt}.$$

Now, if G = const of motion,

$$\partial H = 0.$$

In other words, constants of motion are the generating f's of ICTs that leave H invariant.

For example, if the system is inv wrt a z-axis rotation, an ICT which produces a z-axis rotation will obviously leave H inv. This is a connection between constants of motion & system's symmetries.

Ex. 1. Momentum conservation

Take q_i to be cyclic, and consider $G(q, p, t) = p_i$, then recall that

$$\begin{cases} \delta p_j = -\epsilon \dfrac{\partial G}{\partial q_j} = 0, \\ \delta q_j = \epsilon \dfrac{\partial G}{\partial p_j} = \epsilon \delta_{ij}. \end{cases}$$

Clearly, $\partial H = 0$ under this transform, yielding $\underbrace{\dfrac{dp_i}{dt}}_{\frac{dG}{dt}} = 0 \Rightarrow p_i = \text{const}.$

(2.) Angular momentum

Consider an infinitesimal rot'n of a system by an angle $d\theta_x$ around the z-axis:

$$\begin{cases} \delta x_i = -y_i \, d\theta, \\ \delta y_i = x_i \, d\theta, \\ \delta z_i = 0. \end{cases}$$

Similarly,
$$\begin{cases} \delta p_{i,x} = -p_{i,y} \, d\theta, \\ \delta p_{i,y} = p_{i,x} \, d\theta, \\ \delta p_{i,z} = 0 \end{cases}$$

Define $G = x_i p_{i,y} - y_i p_{i,x}$:

$\epsilon \rightsquigarrow d\theta$

$$\begin{cases} \delta x_i = d\theta \dfrac{\partial G}{\partial p_{i,x}} = -y_i \, d\theta, \\ \delta y_i = d\theta \dfrac{\partial G}{\partial p_{i,y}} = x_i \, d\theta \end{cases} , \quad \delta z_i = 0$$

$$\begin{cases} \delta p_{i,x} = -d\theta \dfrac{\partial G}{\partial x_i} = -p_{i,y} \, d\theta, \\ \delta p_{i,y} = -d\theta \dfrac{\partial G}{\partial y_i} = p_{i,x} \, d\theta \end{cases} , \quad \delta p_{i,z} = 0$$

↑ this indeed creates a rot'n by $d\theta$ around z

Clearly, $G = (\vec{r}_i \times \vec{p}_i)_z = L_z \underset{\times}{=} \vec{L}\cdot\hat{z}$.

More generally, $G = \vec{L}\cdot\vec{n}$, where $\vec{n}$ is the unit vector which defines the axis of rotation.

We can say that $\vec{L}$ is the generator of the spatial rotations of the system.

If $\partial H = 0$ under a spatial rotation, we obtain $\frac{d}{dt}(\vec{L}\cdot\vec{n}) = 0 \Rightarrow \vec{L}\cdot\vec{n} = $ const of motion.