Canonical transformations

Lecture 21

Consider a system for which H is conserved and all q_i's are cyclic.

Then $p_i = \underbrace{\alpha_i}_{\text{const}}$ and $H = \underbrace{H(\alpha_1, \dots, \alpha_n)}_{\text{const}}$

Next, $\dot{q}_i = \frac{\partial H}{\partial \alpha_i} \equiv \underbrace{\omega_i}_{\text{const as well}}$, yielding

$$q_i = \omega_i t + \underbrace{\beta_i}_{\text{const of integration}}$$

In each case (i.e. for each particular system), the number of cyclic coords depends on a choice of generalized coords, and may even be n.

How do we transform from one set of generalized coords to the other?

Previously, $Q_i = \underbrace{Q_i(q, t)}_{\text{point transform, config'n space}}$ $\quad i = 1, \dots, n$

Now, consider $\begin{cases} Q_i = Q_i(q, p, t) \\ P_i = \underbrace{P_i(q, p, t)}_{\text{point transform, phase space}} \end{cases}$

Q_i, P_i must be canonical vars:

$$\dot{Q}_i = \frac{\partial K}{\partial P_i}, \qquad \dot{P}_i = -\frac{\partial K}{\partial Q_i}.$$

Variational principle:

$$\begin{cases} \delta \int_{t_1}^{t_2} dt \,[P_i \dot{Q}_i - K(Q,P,t)] = 0, & \text{new} \\ \delta \int_{t_1}^{t_2} dt \,[p_i \dot{q}_i - H(q,p,t)] = 0 & \text{old} \end{cases}$$

In general, this implies that

$$\underbrace{\lambda}_{\text{const}} (p_i \dot{q}_i - H) = P_i \dot{Q}_i - K + \frac{d\overbrace{F}^{F(P,Q,t)}}{dt}$$

λ is a scale transform:

Consider $\begin{cases} Q_i' = \mu q_i, \\ P_i' = \nu p_i \end{cases} \Rightarrow \underbrace{\nu \dot{p}_i}_{\dot{P}_i'} = -\frac{\partial H}{\partial q_i}\nu = -\nu \frac{\partial H}{\partial (Q_i'/\mu)} = -\nu\mu \frac{\partial H}{\partial Q_i'}.$

Likewise, $\underbrace{\mu \dot{q}_i}_{\dot{Q}_i'} = \mu \frac{\partial H}{\partial \left(\frac{P_i'}{\nu}\right)} = \mu\nu \frac{\partial H}{\partial P_i'}.$

Clearly, $K' = \mu\nu H$ and

$$\underbrace{\mu\nu}_{=\lambda} (p_i \dot{q}_i - H) = P_i' \dot{Q}_i' - K'.$$

So we can focus on $\lambda = 1$ and rescale the coords later if desired. Hence consider

$$p_i \dot{q}_i - H = \underset{\substack{\Uparrow \\ \text{canonical transform}}}{P_i} \dot{Q}_i - K + \frac{dF}{dt} .$$

Restricted canonical transforms:

$$\begin{cases} Q_i = Q_i(q, p), \\ P_i = P_i(q, p) \end{cases}$$ no explicit t-dependence

Note that F can be $F(q, p, t)$ or $F(Q, P, t)$ or a mixture of the two sets of coords.

Suppose that $F = F_1(q, Q, t)$, then

$$p_i \dot{q}_i - H = P_i \dot{Q}_i - K + \frac{dF_1}{dt} =$$

$$= P_i \dot{Q}_i - K + \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial q_i} \dot{q}_i + \frac{\partial F_1}{\partial Q_i} \dot{Q}_i . \qquad (*)$$

Eq'n $(*)$ can hold iff

$$\begin{cases} p_i = \dfrac{\partial F_1}{\partial q_i}, & (1) \\ P_i = -\dfrac{\partial F_1}{\partial Q_i} & (2) \end{cases}$$, yielding

$$K = H + \frac{\partial F_1}{\partial t} . \qquad (3)$$

Eq'n (1) can be inverted to get $Q_i(P, q, t)$, then eq. (2) can be used to find $P_i(p, q, t)$, completing the canonical transformation. Finally, eq'n (3) provides $K(P, Q, t)$ in terms of H & $\frac{\partial F_1}{\partial t}$ expressed in terms of P, Q, t.

Sometimes, other sets of variables are more convenient. For example, consider

$$F = F_2(q, P, t) - Q_i P_i \quad , \quad \text{then}$$

$$p_i \dot{q}_i - H = \cancel{P_i \dot{Q}_i} - K + \frac{dF_2}{dt} - \cancel{\dot{Q}_i P_i} - Q_i \dot{P}_i =$$

$$= - Q_i \dot{P}_i - K + \underbrace{\frac{dF_2}{dt}}_{= \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial q_i} \dot{q}_i + \frac{\partial F_2}{\partial P_i} \dot{P}_i} .$$

This yields

$$\begin{cases} Q_i = \dfrac{\partial F_2}{\partial P_i} , & (1') \\ p_i = \dfrac{\partial F_2}{\partial q_i} , & (2') \\ K = H + \dfrac{\partial F_2}{\partial t} . & (3') \end{cases}$$

Again, (2') can be inverted & (1') used to find $Q_i(p, q, t)$ & $P_i(p, q, t)$ explicitly.

Overall, there are 4 canonical transformations:

$$F = F_1(q, Q, t) \qquad p_i = \frac{\partial F_1}{\partial q_i} \qquad P_i = -\frac{\partial F_1}{\partial Q_i}$$

$$F = F_2(q, P, t) - Q_i P_i \qquad p_i = \frac{\partial F_2}{\partial q_i} \qquad Q_i = \frac{\partial F_2}{\partial P_i}$$

$$F = F_3(p, Q, t) + q_i p_i \qquad q_i = -\frac{\partial F_3}{\partial p_i} \qquad P_i = -\frac{\partial F_3}{\partial Q_i}$$

$$F = F_4(p, P, t) + q_i p_i - Q_i P_i \qquad q_i = -\frac{\partial F_4}{\partial p_i} \qquad Q_i = \frac{\partial F_4}{\partial P_i}$$

F is the generating function of the canonical transform

Note that the generating f's can be mixed: e.g. $F'(q_1, p_2, P_1, Q_2, t)$ for $n=2$

Then $F = F' - Q_1 P_1 + q_2 p_2$, and

$$\begin{cases} p_1 = \dfrac{\partial F'}{\partial q_1} & Q_1 = \dfrac{\partial F'}{\partial P_1} \\[2ex] q_2 = -\dfrac{\partial F'}{\partial p_2} & P_2 = -\dfrac{\partial F'}{\partial Q_2} \end{cases}$$

$$K = H + \frac{\partial F'}{\partial t}$$

Indeed, $p_i \dot{q}_i - H = P_i \dot{Q}_i - K + \dot{q}_2 p_2 + \dot{p}_2 q_2 - \dot{Q}_1 P_1 - Q_1 \dot{P}_1 + \frac{\partial F'}{\partial t} + \frac{\partial F'}{\partial q_1}\dot{q}_1 + \frac{\partial F'}{\partial p_2}\dot{p}_2 + \frac{\partial F'}{\partial P_1}\dot{P}_1 + \frac{\partial F'}{\partial Q_2}\dot{Q}_2$,

$$p_1 \dot{q}_1 - H = P_2 \dot{Q}_2 - K + \dot{p}_2 q_2 - Q_1 \dot{P}_1 + \frac{\partial F'}{\partial t} + \frac{\partial F'}{\partial p_2}\dot{p}_2 + \frac{\partial F'}{\partial P_1}\dot{P}_1 + \frac{\partial F'}{\partial Q_2}\dot{Q}_2 + \frac{\partial F'}{\partial q_1}\dot{q}_1$$

Examples

1. Consider $F_2 = q_i P_i$, then

$$p_i = P_i \,, \quad Q_i = q_i \,, \quad K = H$$

identity transformation

2. More generally, consider

$$F_2 = f_i(q_1, \dots, q_n; t) P_i \,, \quad \text{then}$$

$$\underbrace{Q_i = \frac{\partial F_2}{\partial P_i} = f_i(q;t)}_{\text{can be inverted}} \quad \text{to get } q_i(Q;t)$$

Defines a class of canonical point transformations which involve coords only.

Moreover, $p_j = P_i \dfrac{\partial f_i}{\partial q_j}$ and $K = H + P_i \dfrac{\partial f_i}{\partial t}$

3. Even more generally, consider

$$F_2 = f_i(q;t) P_i + g(q;t) .$$

Then $Q_i = f_i(q;t)$ again, but

$$p_j = \frac{\partial F_2}{\partial q_j} = P_i \frac{\partial f_i}{\partial q_j} + \frac{\partial g}{\partial q_j} \,, \quad \text{or}$$

for $n=2$

$$\underbrace{\begin{pmatrix} p_1 \\ p_2 \end{pmatrix}}_{\vec{p}} = \underbrace{\begin{pmatrix} \frac{\partial f_1}{\partial q_1} & \frac{\partial f_2}{\partial q_1} \\ \frac{\partial f_1}{\partial q_2} & \frac{\partial f_2}{\partial q_2} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} P_1 \\ P_2 \end{pmatrix}}_{\vec{P}} + \underbrace{\begin{pmatrix} \frac{\partial g}{\partial q_1} \\ \frac{\partial g}{\partial q_2} \end{pmatrix}}_{\vec{b}} \quad , \text{ or}$$

$$\vec{P} = A^{-1}(\vec{p} - \vec{b}) .$$

4. Finally, consider

$$F_1 = q_j Q_j .$$

Then $\begin{cases} p_i = \dfrac{\partial F_1}{\partial q_i} = Q_i , \\ P_i = -\dfrac{\partial F_1}{\partial Q_i} = -q_i . \end{cases}$ coords & momenta exchanged

5. Mixed transform:

$$F' = q_1 P_1 + q_2 Q_2 \quad \text{yields}$$

$$\begin{aligned} Q_1 &= q_1 & P_1 &= p_1 \\ Q_2 &= p_2 & P_2 &= -q_2 \end{aligned} \quad \Leftarrow \text{ only } p_2 \,\&\, q_2 \text{ are swapped}$$

Harmonic oscillator

Consider $H = \dfrac{p^2}{2m} + \dfrac{kq^2}{2} \underset{k = m\omega^2}{=} \dfrac{1}{2m}\left(p^2 + m^2\omega^2 q^2\right)$. $(*)$

Sum of squares in $(*)$ suggests trying

$$\begin{cases} p = f(P)\cos Q, \\ m\omega q = f(P)\sin Q \end{cases} \Rightarrow H = \frac{f^2(P)}{2m}$$

and Q is cyclic.

Need to find $f(P)$:

try $F_1(q, Q) = \dfrac{m\omega q^2}{2}\cot Q$,

then

$$\begin{cases} p = \dfrac{\partial F_1}{\partial q} = m\omega q \cot Q, \\ P = -\dfrac{\partial F_1}{\partial Q} = -\dfrac{m\omega q^2}{2} \times \left(-\dfrac{\sin Q}{\sin Q} - \cos Q \dfrac{1}{\sin^2 Q} \times \cos Q\right) = \dfrac{m\omega q^2}{2\sin^2 Q}. \end{cases}$$

$$-1 - \frac{\cos^2 Q}{\sin^2 Q} = -\frac{1}{\sin^2 Q}$$

Then

$$\begin{cases} q = \sqrt{\dfrac{2P}{m\omega}}\sin Q, \\ p = \underbrace{\sqrt{2m\omega P}}_{f(P)} \underbrace{\cot Q \sin Q}_{\cos Q} \end{cases}$$

So, $f(P) = \sqrt{2m\omega P}$ and

$$H = \omega \underline{\underline{P}}. \qquad \dot{P} = -\frac{\partial H}{\partial Q} = 0$$

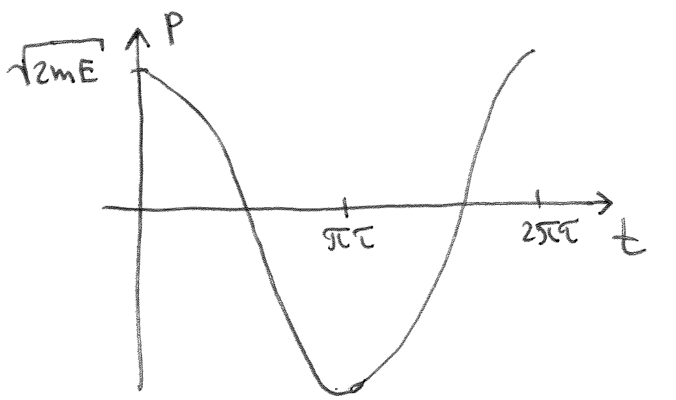
Note that $P = \text{const}$ since Q is cyclic.

In fact, $P = \dfrac{E}{\omega}$ ← total energy

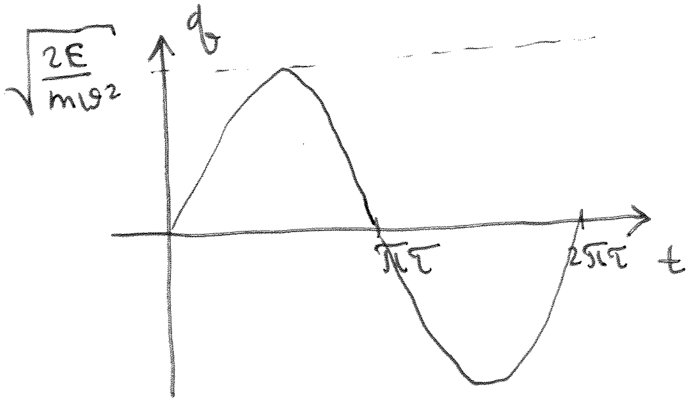
EoM: $\dot{Q} = \dfrac{\partial H}{\partial P} = \omega \Rightarrow Q = \omega t + \alpha$

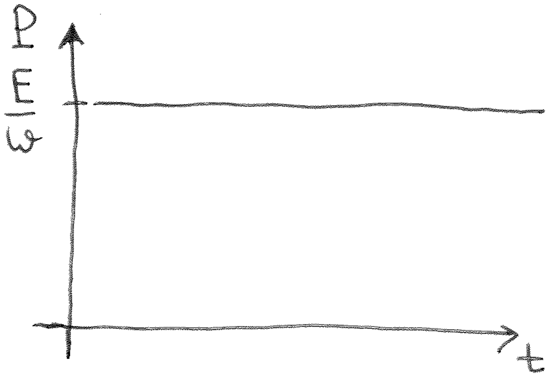
Then
$$\begin{cases} q(t) = \sqrt{\dfrac{2E}{m\omega^2}} \sin(\omega t + \alpha), \\ p(t) = \sqrt{2mE} \cos(\omega t + \alpha). \end{cases}$$

⇐ can be confirmed by solving EoMs directly



⇐ $\alpha = 0$ for simplicity (in all plots)

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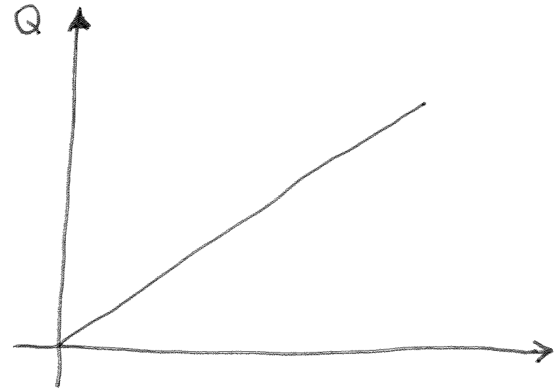
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