Forced oscillations and the effect of dissipative forces

Lecture 18

System is acted upon by an external driving force that acts at $t \geq 0$.

If F_j is the generalized force corresponding to η_j, we have:

$$\underbrace{Q_j}_{\substack{\text{generalized} \\ \text{force for} \\ \xi_j}} = \sum_i F_i \underbrace{\frac{\partial \eta_i}{\partial \xi_j}}_{a_{ij}}$$

a_{ij} since $\eta_i = a_{ij} \xi_j$

Then $\ddot{\xi}_i + \omega_i^2 \xi_i = Q_i \quad (*) \quad i = 1, \dots, n$

Frequently, the driving force is periodic: $Q_i = Q_{0i} \cos(\omega t + \delta_i)$

In general, the solutions of Eq. (*) are a combination of free oscillations (sol'n to a homog. eq'n) and a particular solution of the inhomogeneous eq'n.

Focusing on the latter, consider

$\xi_i = B_i \cos(\omega t + \delta_i)$:

$-\omega^2 B_i + \omega_i^2 B_i = Q_{0i}$, or

$$B_i = \frac{Q_{0i}}{\omega_i^2 - \omega^2}.$$

Then $\underbrace{\eta_j = a_{ji} \xi_i}_{\text{sum over } i} = \underbrace{\frac{a_{ji} Q_{0i} \cos(\omega t + \delta_i)}{\omega_i^2 - \omega^2}}_{\substack{\text{linear combination} \\ \text{of normal modes}}}$

Clearly, the amplitude of each η_j will depend on Q_{0i}'s but also on

$\frac{1}{\omega_i^2 - \omega^2}$ $\Leftarrow$ resonance.

In fact, the theory breaks down if $\omega \simeq \omega_i$ for some i because the oscillations are no longer small.

In most realistic systems, there are friction or dissipative forces, which are typically $\sim \dot{\eta}_i$. Then we can introduce

$$\underset{\substack{\uparrow \\ \text{dissipation f'n}}}{\mathcal{F}} = \frac{1}{2} \mathcal{F}_{ij} \dot{\eta}_i \dot{\eta}_j \qquad \mathcal{F}_{ij} = \mathcal{F}_{ji}$$

As with T, $\mathcal{F}_{ij}$'s are constants to $\mathcal{O}(\dot{\eta}_i^2)$, rather than f's of η_i.

Recall that $\frac{dE}{dt} = -2 \underset{>0}{\mathcal{F}}$

EoM:

$$T_{ij}\ddot{\eta}_j + \mathcal{F}_{ij}\dot{\eta}_j + V_{ij}\eta_j = 0$$

In general, it's impossible to diagonalize T, $\mathcal{F}$, V simultaneously.

However, often enough T & $\mathcal{F}$ are diagonal already, in which case we can rescale $T \Rightarrow \mathbb{I}$ (i.e. apply $\tilde{A}TA = \mathbb{I}$), which yields

$$\mathcal{F} = \frac{1}{2}\dot{\xi}^{+}\underbrace{(\tilde{A}\mathcal{F}A)}_{\mathcal{F}'}\dot{\xi}$$

$\mathcal{F}' \Leftarrow$ diagonal since $\mathcal{F}$ was diagonal

diagonal elements of $\mathcal{F}'$, must be ≥ 0

Then $$\ddot{\xi}_i + \mathcal{F}_i\dot{\xi}_i + \omega_i^2\xi_i = 0 \qquad (**)$$

(no sum over i)

Eq'n (**) can be solved by

$$\xi_i = C_i e^{-i\omega_i' t}:$$

$$\omega_i'^2 + i\omega_i'\mathcal{F}_i - \omega_i^2 = 0,$$ solved by

$$(\omega_i')_{1,2} = \frac{-i\mathcal{F}_i \pm \sqrt{-\mathcal{F}_i^2 + 4\omega_i^2}}{2} =$$

$$= \underbrace{\pm\sqrt{\omega_i^2 - \frac{\mathcal{F}_i^2}{4}}}_{\text{vibration freqs are shifted}} \underbrace{- i\frac{\mathcal{F}_i}{2}}_{\text{complex part}}$$

So, ω_i' is complex:

$$\xi_i \sim e^{-\frac{F_i t}{2}}, \quad \text{damped oscillation}$$

If dissipation is small, $\omega_i^2 - \frac{F_i^2}{4} \approx \omega_i^2$

and $$\xi_i \simeq C_i e^{-\frac{F_i t}{2}} e^{-i\omega_i t}.$$

Even if T & F are not diagonal, we can work directly with

$$T_{ij}\ddot{\eta}_j + F_{ij}\dot{\eta}_j + V_{ij}\eta_j = 0 \quad \text{as follows:}$$

try $\eta_j = a_j e^{-i\omega t} = a_j e^{-\kappa t} e^{-2\pi i \nu t}$, then

$$-i\omega \equiv \gamma = -\kappa - i \times 2\pi\nu$$

$$V_{ij} a_j \underset{\uparrow \gamma}{-i\omega} F_{ij} a_j - \underset{\uparrow \gamma^2}{\omega^2} T_{ij} a_j = 0, \text{ or}$$

$$\underline{\underline{V\vec{a} + \gamma F\vec{a} + \gamma^2 T\vec{a} = 0}}.$$

Without solving this eq'n directly, consider

$$\underbrace{\vec{a}^{+} V \vec{a}}_{a} + \gamma \underbrace{\vec{a}^{+} F \vec{a}}_{b} + \gamma^2 \underbrace{\vec{a}^{+} T \vec{a}}_{c} = 0$$

a, b, c are all Re since V, F, T are symmetric (as discussed above) and shown below

Then

$$\gamma_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2c} \quad , \text{or}$$

$$\gamma_1 + \gamma_2 = -\frac{b}{c}.$$

However, if γ is a solution, γ^* is also a solution of the same quadratic eq'n:

$$c(\gamma^*)^2 + b\gamma^* + a = 0 \quad \Rightarrow \quad \begin{cases} \gamma_1 = \gamma, \\ \gamma_2 = \gamma^* \end{cases}$$

note that
$(\gamma^2)^* = (\gamma^*)^2$

So, $\gamma + \gamma^* = -2K = -\dfrac{\vec{a}^+ F \vec{a}}{\vec{a}^+ T \vec{a}}$, or

$a_j = \alpha_j + i\beta_j$

$$K = \frac{1}{2} \frac{(\alpha_i - i\beta_i) F_{ij} (\alpha_j + i\beta_j)}{(\alpha_i - i\beta_i) T_{ij} (\alpha_j + i\beta_j)} \;\textcircled{=}$$

$$-i\beta_i \overbrace{F_{ij}}^{F_{ji}} \alpha_j + \alpha_i F_{ij}(i\beta_j) =$$

$$= -i\alpha_j F_{ji}\beta_i + i\alpha_i F_{ij}\beta_j = 0 \quad , \text{and same for } T$$

$$\textcircled{=} \; \frac{1}{2} \frac{F_{ij}(\alpha_i\alpha_j + \beta_i\beta_j)}{T_{ij}(\alpha_i\alpha_j + \beta_i\beta_j)}$$

Since $\begin{cases} \alpha_i F_{ij} \alpha_j \geq 0 \\ \alpha_i T_{ij} \alpha_j > 0 \end{cases}$ for each "velocity" $\vec{\alpha}$,

$$\underline{\underline{K \geq 0}}.$$

Finally, consider

$$\underbrace{F_j}_{\text{periodic driving force}} = \overset{\text{complex in general}}{\tilde{F}_{0j}}\, e^{-i\omega t}$$

<u>EoM</u>: $$T_{ij}\ddot{\eta}_j + \mathcal{F}_{ij}\dot{\eta}_j + V_{ij}\eta_j = F_{0i}\, e^{-i\omega t}.$$

Try $\underline{\eta_j = A_j e^{-i\omega t}}$ as a particular solution:

$$\Rightarrow \left(V_{ij} - i\omega \mathcal{F}_{ij} - \omega^2 T_{ij}\right) A_j = F_{0i}$$

n linear inhomog. eq's, solved using Cramer's rule:

$$A_j = \frac{D_j(\omega)}{D(\omega)}$$

$D(\omega)$ = $n\times n$ determinant of the A_j coefficients, e.g. in the 2×2 case:

$$D(\omega) = \begin{vmatrix} V_{11} - i\omega\mathcal{F}_{11} - \omega^2 T_{11} & V_{12} - i\omega\mathcal{F}_{12} - \omega^2 T_{12} \\ V_{21} - i\omega\mathcal{F}_{21} - \omega^2 T_{21} & V_{22} - i\omega\mathcal{F}_{22} - \omega^2 T_{22} \end{vmatrix}$$

$D_j(\omega)$ = $D(\omega)$ with j^{th} column replaced by $\begin{pmatrix} F_{01} \\ \vdots \\ F_{0n} \end{pmatrix}$. In the 2×2 case,

$$D_1(\omega) = \begin{vmatrix} F_{01} & V_{12} - i\omega\mathcal{F}_{12} - \omega^2 T_{12} \\ F_{02} & V_{22} - i\omega\mathcal{F}_{22} - \omega^2 T_{22} \end{vmatrix}$$ & similarly for $D_2(\omega)$

Focusing on $D(\omega)$, we observe that it is exactly the determinant of the corresponding force-free system of eq's:

$$(V_{ij} - i\omega F_{ij} - \omega^2 T_{ij}) A_j = 0$$

$D(\omega) = 0$ is therefore an eq. of order $2n$ for ω.

However, note that, similar to the argument about γ & γ^*, if ω_j is a solution, so is $-\omega_j^*$. So, the solutions are $\begin{cases} \omega_1 \ldots \omega_n \\ -\omega_1^* \ldots -\omega_n^* \end{cases}$

$\left\{ \begin{matrix} \gamma = -i\omega \text{ is a solution, and} \\ \gamma^* = i\omega^* \text{ is also a solution} \end{matrix} \right\}$

This implies that

$$D(\omega) = \underset{\text{const}}{G}\,(\omega - \omega_1)(\omega - \omega_2)\ldots(\omega - \omega_n)(\omega + \omega_1^*)\ldots(\omega + \omega_n^*) \boxed{\equiv}$$

$\omega_j = -iK_j + 2\pi\nu_j$ [also, use $\omega = 2\pi\nu$]

$$\boxed{\equiv}\; G \prod_{i=1}^{n} \left[(2\pi(\nu - \nu_i) + iK_i)(2\pi(\nu + \nu_i) + iK_i) \right],$$

so that

$$\underbrace{D(\omega) D^*(\omega)}_{} = G G^* \prod_{i=1}^{n} \left[(4\pi^2(\nu - \nu_i)^2 + K_i^2)(4\pi^2(\nu + \nu_i)^2 + K_i^2) \right]$$

this will appear when we try to find $\underbrace{Re\{A_j\}}_{\equiv a_j}$ & $\underbrace{Im\{A_j\}}_{\equiv b_j}$:

$$Re\{\eta_j\} = Re\{(a_j + ib_j)(\cos\omega t - i\sin\omega t)\} =$$

$$= a_j \cos\omega t + b_j \sin\omega t$$

To find a_j & b_j, use

$$a_j + i b_j = \frac{\overbrace{D_j(\omega)}^{\equiv c_j + i d_j}}{\underbrace{D(\omega)}_{= c + i d}} = \frac{(c_j + i d_j)(c - i d)}{\underbrace{c^2 + d^2}_{= D(\omega) D^*(\omega)}} .$$

So,
$$\begin{cases} a_j = \dfrac{c_j c + d_j d}{D(\omega) D^*(\omega)}, \\ b_j = \dfrac{d_j c - c_j d}{D(\omega) D^*(\omega)} \end{cases} \quad (*)$$

Clearly, $\omega = \pm \omega_j$ represent resonant frequencies but the amplitudes remain finite since $K_i \neq 0$.

The numerators in $(*)$ are also ω-dependent, leading to (small) shifts in resonance freqs away from $\pm \omega_j$.