Fast top

Now focus on the $\theta=\theta_0$, $\dot\theta=0$, $\dot\varphi=0$ initial conditions and in addition stipulate

$$\underbrace{\frac{I_3\omega_3^2}{2}}_{\substack{\text{kinetic}\\ \text{energy of}\\ \text{rot'n around}\\ \text{z-axis}}} \gg \underbrace{Mgl}_{\substack{\text{max change}\\ \text{in } V}} \qquad \Leftarrow \text{"fast top"}$$

Then, since $u_0=u_2$, the extent of nutation is given by $|u_1-u_0|$.

$$E' = Mgl\overbrace{\cos\theta_0}^{u_0} \Rightarrow \alpha = \frac{2E'}{I_1} = \underbrace{\frac{2Mgl}{I_1}}_{\beta} u_0 = \beta u_0$$

$E' = E - \frac{I_3\omega_3^2}{2}$

$\Uparrow$ since $\dot\varphi=0$, $\dot\theta=0$ at $t=0$

Next, $b=u_0 a$, $\alpha = \beta u_0$

$$f(u) = \beta u^3 - (\underbrace{\alpha}_{\beta u_0}+a^2)u^2 + (2ab-\beta)u + (\underbrace{\alpha}_{\beta u_0}-b^2) =$$

$$= \beta u^2(u-u_0) - a^2u^2 - \beta(u-u_0) + 2u_0a^2u - u_0^2a^2 =$$

$$= (u_0-u)\beta(1-u^2) - a^2\underbrace{(u^2-2u_0u+u_0^2)}_{(u-u_0)^2} =$$

$$= (u_0-u)\left[\beta(1-u^2) - a^2(u_0-u)\right]$$

One root is u_0 as discussed above, and the other two are given by

$$(1-u^2) - \frac{a^2}{\beta}(u_0 - u) = 0 .$$

Define $x = u_0 - u$: $u^2 = (u_0 - x)^2 = u_0^2 - 2u_0 x + x^2$, and

$$1 - u_0^2 + \underline{2u_0 x} - x^2 - \underline{\frac{a^2}{\beta} x} = 0 , \text{ or}$$

$$x^2 + x\underbrace{\left(\frac{a^2}{\beta} - 2u_0\right)}_{\text{"}p} + \underbrace{u_0^2 - 1}_{-q,\text{ where}} = 0 .$$

$q = 1 - \cos^2\theta_0 = \sin^2\theta_0$

"Fast top" implies that $p \gg q$. Indeed,

$$\frac{a^2}{\beta} = \frac{p_\psi^2}{2Mgl I_1} \overset{p_\psi = I_3\omega_3}{=} \underbrace{\left(\frac{I_3}{I_1}\right)}_{\theta(1)\text{ unless}} \underbrace{\frac{I_3 \omega_3^2}{2Mgl}}_{\gg 1 \text{ for fast top}} \gg 1 .$$

$I_3 \ll I_1$ (we ignore this possibility)

Then $p \approx \overbrace{\frac{a^2}{\beta}}^{\gg 1} \gg \underbrace{q}_{\theta(1)}$, and

$$x_{1,2} = \frac{-p \pm \sqrt{p^2 + 4q}}{2} = -\frac{p}{2} \pm \frac{p}{2}\sqrt{1 + 4\frac{q}{p^2}} \approx$$

$$\approx -\frac{p}{2} \pm \frac{p}{2}\left(1 + \frac{2q}{p^2}\right).$$

The "$-$" solution is $\simeq -p \ll -1$, unphysical [same as the $u > 1$ solution from before]

So, only the "+" solution remains:

$$x_1 = \frac{\alpha}{p} \simeq \frac{\beta \sin^2\theta_0}{a^2} = \left(\frac{I_1}{I_3}\right) \frac{2Mgl}{I_3 \omega_3^2} \sin^2\theta_0 .$$

$x_1 = u_0 - u_1$

Note that $x_1 \sim \frac{1}{\omega_3^2}$ $\Leftarrow$ nutation $\searrow$ as the top spins up

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Further, if the extent of nutation is small, $1-u^2 \simeq \sin^2\theta_0$, and

$$\underbrace{f(u)}_{=\dot{u}^2 = \dot{x}^2} \simeq x\left[\beta \sin^2\theta_0 - a^2 x\right] = a^2 x\left[x_1 - x\right].$$

Next, define $y = x - \frac{x_1}{2}$ s.t.

$$\dot{y}^2 = \left(y + \frac{x_1}{2}\right) a^2 \left(x_1 - y - \frac{x_1}{2}\right) = a^2\left(\frac{x_1^2}{4} - y^2\right).$$

Differentiate: $2\dot{y}\ddot{y} = -2a^2 y \dot{y}$, or

$$\boxed{\ddot{y} = -a^2 y} \quad (*)$$

also, $\dot{y}|_{t=0} = 0$, consistent with $\dot{\theta}|_{t=0} = 0$

$(*)$ is solved by $y(t) = -\frac{x_1}{2}\cos(at)$, or

$$x(t) = \frac{x_1}{2}(1 - \cos(at))$$ $\Leftarrow$ note that $x(0) = 0$ as desired

So, the freq. of nutation between θ_0 & θ_1 is

$$a = \frac{I_3 \omega_3}{I_1} \sim \omega_3 \quad (\text{i.e. } a \uparrow \text{ as } \omega_3 \uparrow)$$

Finally,

$$\dot{\varphi} = \frac{b - a\cos\theta}{\sin^2\theta} \overset{b = au_0}{\simeq} \frac{a(u_0 - u)}{\sin^2\theta_0} = \frac{ax}{\sin^2\theta_0}, \text{ yielding}$$

$$\dot{\varphi} = \frac{a}{\sin^2\theta_0} \underbrace{\frac{x_1}{2}}_{\frac{\beta \sin^2\theta_0}{2a^2}} (1 - \cos(at)) = \frac{\beta}{2a}(1 - \cos(at))$$

Freq. of precession is the same as freq. of nutation (a).

$$\langle \dot{\varphi} \rangle = \frac{\beta}{2a} = \frac{Mgl}{I_3 \omega_3} \sim \frac{1}{\omega_3}.$$

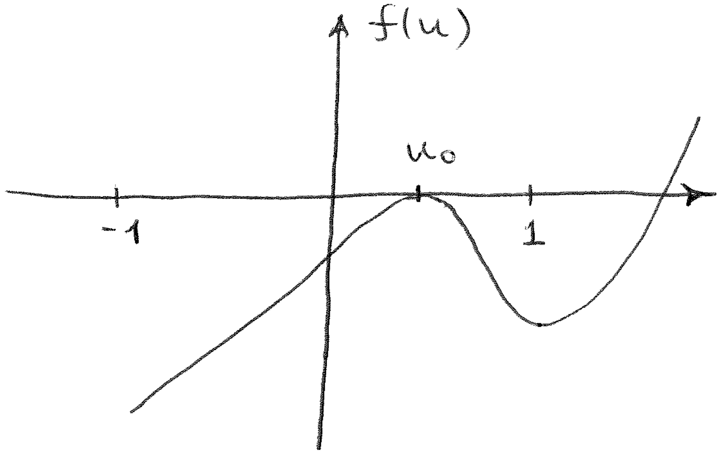
The rate of precession $\downarrow$ as $\omega_3 \uparrow$.

This is <u>pseudoregular</u> precession since nutation is always present (but may be masked by friction at the pivot if it becomes too small).

Is it possible to choose initial conditions to create <u>true</u> regular precession?

$$\theta_1 = \theta_2 = \underbrace{\theta_0}_{\text{initial value } (t=0)}$$

We must have:



at $u = u_0$:

$$\begin{cases} f(u) = \dot{u}^2 = 0, \\ \dfrac{df}{du} = 0 \end{cases}$$

Then $(1-u_0^2)(\alpha - \beta u_0) - (b - a u_0)^2 = 0$, or

$$\alpha - \beta u_0 = \frac{(b - a u_0)^2}{1 - u_0^2}$$

$$\left.\frac{df}{du}\right|_{u_0} = 0 \Rightarrow (-2u_0)(\alpha - \beta u_0) + (1 - u_0^2)(-\beta) - $$
$$- 2(b - a u_0)(-a) = 0, \text{ or}$$

$$\frac{\beta}{2}(1 - u_0^2) = a(b - a u_0) - u_0(\alpha - \beta u_0).$$

So, $$\frac{\beta}{2} = a \underbrace{\frac{b - a u_0}{1 - u_0^2}}_{\dot{\varphi}} - \underbrace{u_0 \frac{(b - a u_0)^2}{(1 - u_0^2)^2}}_{\dot{\varphi}^2 \times u_0} \ominus$$

[note that $\dot{\psi} = \text{const}$, $\dot{\varphi} = \text{const}$ here]

$$\ominus\ a\dot{\varphi} - \dot{\varphi}^2 \cos\theta_0.$$

Now, $$\underbrace{Mgl}_{\text{comes from } \beta} = I_1 \dot{\varphi}(\underbrace{a}_{\frac{I_3 \omega_3}{I_1}} - \dot{\varphi}\cos\theta_0) \quad (***) \quad \boxed{\equiv}$$

$\boxed{=}$ $\dot{\varphi}(I_3 \omega_3 - I_1 \dot{\varphi} \cos\theta_0)$, or

$$Mgl = \dot{\varphi}\left[I_3 \dot{\psi} + I_3 \dot{\varphi} \cos\theta_0 - I_1 \dot{\varphi} \cos\theta_0\right] =$$

$$= \dot{\varphi}\left[I_3 \dot{\psi} - (I_1 - I_3)\dot{\varphi} \cos\theta_0\right]. \quad (**)$$

Eq'n (**) must be satisfied by the initial conditions. In general, we get to choose $\theta, \varphi, \psi, \underset{\substack{\| \\ 0 \\ \text{here}}}{\dot{\theta}}, \dot{\varphi}, \underbrace{\dot{\psi}}_{\text{or } \omega_3}$ at $t=0$.

Many sets of values will work (for example, init. values of φ & ψ are largely irrelevant). However, note that

$$I_1 \cos\theta_0 \, \dot{\varphi}^2 - \overset{\text{from } (**)}{I_3 \omega_3 \dot{\varphi}} + Mgl = 0$$

requires

$$\underbrace{I_3^2 \omega_3^2 - 4 I_1 \cos\theta_0 \cdot Mgl}_{\text{discriminant}} > 0$$

discriminant, otherwise $\dot{\varphi}$ is not real

If $0 < \theta_0 < \frac{\pi}{2}$ (i.e. $\cos\theta_0 > 0$), we must have

$$\omega_3 > \omega_3^{min} = \frac{2}{I_3}\sqrt{Mgl\, I_1 \cos\theta_0} \quad (***)$$

ω_3 & θ_0 values must satisfy (***)

Since (**) is a quadratic equation, we obtain 2 solutions for $\dot{\varphi}$ in general, "fast" and "slow"! Note that $\dot{\varphi} = 0$ is <u>not</u> a solution $\Rightarrow$ you need a little initial push to set up regular precession.

For "slow" precession,

$$a - \dot{\varphi}\cos\theta_0 \simeq a, \text{ leading to}$$

$$\dot{\varphi} \approx \underline{\underline{\frac{Mgl}{I_3 \omega_3}}} \quad \text{(slow)}$$

This is the same as $\langle \dot{\varphi} \rangle$ for pseudoregular precession.

For "fast" precession, Mgl can be neglected, and

$$\dot{\varphi} \approx \frac{I_3 \omega_3}{I_1 \cos\theta_0} \quad \text{(fast)} \quad \text{indep. of } g \text{ (!)}$$

Another special case:

$\underbrace{u=1}_{\text{initially vertical axis}}$ is one of the roots of $f(u)$

Here, $a = b$ since z & z' axes initially coincide.

$\Uparrow$ $p_\varphi = \underbrace{\text{const}}_{I_1 b}$, $p_\psi = \underbrace{\text{const}}_{I_1 a}$ & $p_\varphi = p_\psi$ at $t=0$

Further,

$$E' = E - \frac{I_3 \omega_3^2}{2} = Mgl \,, \qquad (\cos\theta_0 = 1)$$

s.t. $\quad \alpha = \frac{2E'}{I_1} = \frac{2Mgl}{I_1} = \beta .$

But then

$$\dot{u}^2 = \beta(1-u)(1-u^2) - a^2(1-u)^2 \,, \text{ or}$$

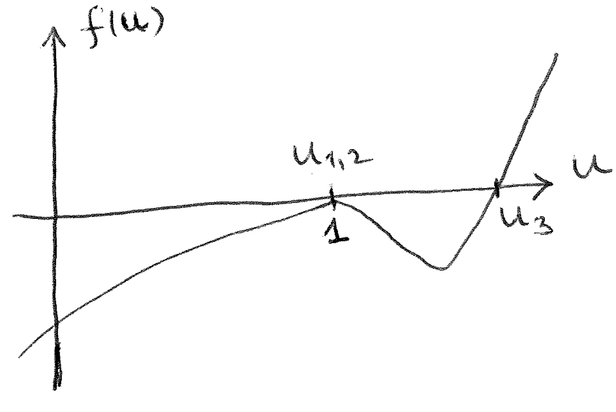
$$\dot{u}^2 = \underbrace{(1-u)^2}_{u_{1,2}=1 \text{ is a double root,}} \underline{\underline{\left[\, \beta(1+u) - a^2 \,\right]}} .$$

$$u_3 = \frac{a^2}{\beta} - 1 .$$

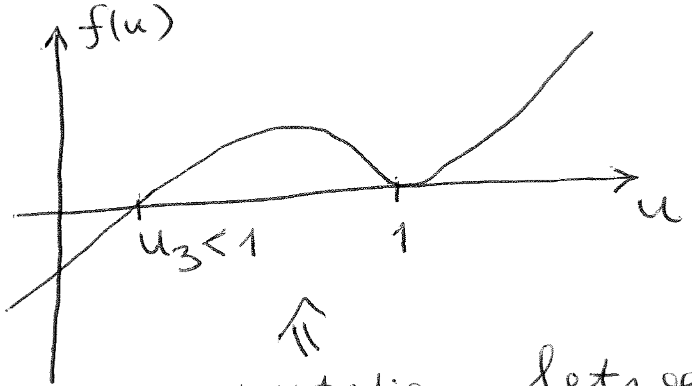
If $\frac{a^2}{\beta} > 2$, ("fast" top)

$u_3 > 1$ & the only motion is $u = 1$ $(\theta = 0)$ $\Rightarrow$

$\Rightarrow$ the top continues to spin about the vertical axis.



If $\frac{\alpha^2}{\beta} < 2$, we have:



⇑ nutation between $\theta = 0$ & $\theta = \theta_3$

Critical value:

$$\frac{\alpha^2}{\beta} = \left(\frac{I_3}{I_1}\right)\frac{I_3\,\omega^2_{crit}}{2Mgl} = 2, \quad \text{yielding}$$

$$\omega^2_{crit} = 4\,\frac{Mgl\,I_1}{I_3^2}.$$

Note that $\omega^2_{crit} = \underbrace{(\omega_3^{min})^2}_{\text{min freq.}}\Big|_{\theta_0=0}$, as expected

min freq. of uniform precession from discriminant analysis (Eq. (***))

In practice, if $\omega_3 > \omega_{crit}$ at $t=0$ (and with $\theta_0 = 0$), the top will stay vertical until friction slows it down to ω_{crit}, at which point it begins to wobble more and more.