The Coriolis effect

Lecture 12

Apply $\left(\frac{d}{dt}\right)_{lab} = \left(\frac{d}{dt}\right)_{body} + \vec{\omega}\times$

to $\vec{r}$: $\leftarrow$ radius vector of a particle

$$\vec{v}_{lab} = \vec{v}_{body} + \vec{\omega}\times\underbrace{\vec{r}}_{\text{defined in lab frame, say}}$$

$\vec{\omega} = \text{const}$ here

Next, $\vec{a}_{lab} = \left(\frac{d\vec{v}_{lab}}{dt}\right)_{lab} = \left(\frac{d\vec{v}_{lab}}{dt}\right)_{body} + \vec{\omega}\times\vec{v}_{lab} =$

$$= \underbrace{\left(\frac{d\vec{v}_{body}}{dt}\right)_{body}}_{\vec{a}_{body}} + \vec{\omega}\times\underbrace{\left(\frac{d\vec{r}}{dt}\right)_{body}}_{\vec{v}_{body}} + \vec{\omega}\times\vec{v}_{body} + \vec{\omega}\times(\vec{\omega}\times\vec{r}) \; \boxed{=}$$

$$\boxed{=} \; \vec{a}_{body} + 2(\vec{\omega}\times\vec{v}_{body}) + \vec{\omega}\times(\vec{\omega}\times\vec{r}).$$

Finally, the EoM is

$$\vec{F} = m\vec{a}_{lab}, \Leftarrow \text{inertial frame}$$

giving

$$\underbrace{\vec{F} - 2m(\vec{\omega}\times\vec{v}_{Body}) - \underbrace{m\,\vec{\omega}\times(\vec{\omega}\times\vec{r})}_{\text{centrifugal force}}}_{\vec{F}_{eff}} = m\vec{a}_{body}$$

If $\vec{v}_{body} = 0$ (particle stationary in body frame), centrifugal force is the only correction. The $2m(\vec{\omega} \times \vec{v}_{body})$ term is the effective Coriolis force, due to using the non-inertial frame.

Note that

ccw rotation

$$\omega_{Earth} \simeq 7.3 \times 10^{-5}\ s^{-1}, \text{ s.t.}$$

$$\text{centrifugal: } \omega^2_{Earth} \underbrace{R_{Earth}}_{\simeq 6.36 \times 10^3\ km} \simeq 3.4\ \frac{cm}{s^2} \simeq \underline{\underline{3 \times 10^{-3}\ g}}.$$

$$\text{Coriolis: } 2\omega_{Earth} v \simeq 1.5 \times 10^{-4} \underbrace{v}_{\text{in } \frac{cm}{s}}\ \frac{cm}{s^2}$$

If $v = 10^5 \frac{cm}{s} = 1 \frac{km}{s}$,

$$2\omega_{Earth} v \simeq 15\ \frac{cm}{s^2} \simeq 1.5 \times 10^{-2}\ g$$

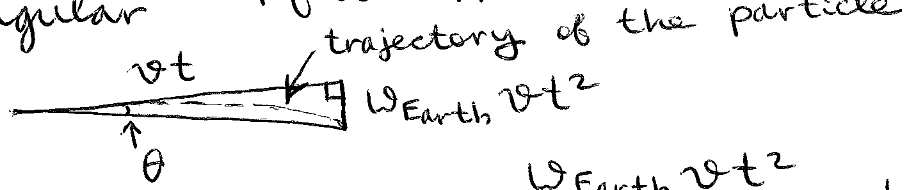
Note also that $\vec{F}_{cor} = -2m(\vec{\omega}_{Earth} \times \vec{v})$ in Earth frame is $\perp \vec{\omega}_{Earth}$, $\perp \vec{v}$, and $= 0$ if $\underbrace{\vec{\omega}_{Earth} \times \vec{v} = 0}_{\vec{\omega}_{Earth} \parallel \vec{v}}$

For ex., a particle fired vertically at the North pole is not affected, but if the particle is fired horizontally,

$$\frac{|F_{cor}|}{m} = 2\omega_{Earth} v.$$

Linear deflection: $\frac{1}{2}(2\omega_{Earth} v)\underbrace{}_{t^2} = \omega_{Earth} v t^2$.

Angular deflection:



$$\sin\theta \simeq \theta = \frac{\omega_{Earth} v t^2}{v t} = \omega_{Earth} t,$$

as expected

The trajectory in the inertial frame is a straight line, with apparent deflection due to Earth rotating beneath the projectile.

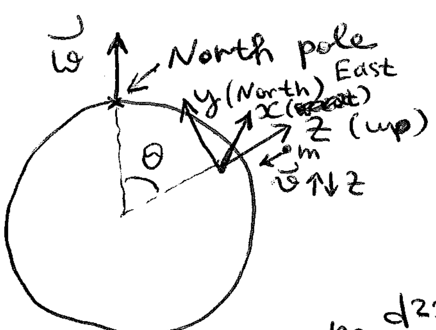
For ex., if $t = 10^2$ s $\Rightarrow \omega_{Earth} t \simeq 7\times 10^{-3}$ rad $\simeq$ $\simeq 0.4°$.

Another example: deflection of a freely falling particle.

$\vec{F}_{cor}$ is in the East-West direction (to the East in the northern hemisphere)

$\Uparrow$ indeed,

$\vec{v}\times\vec{\omega} \uparrow\uparrow x$



Then

$$m\frac{d^2x}{dt^2} = 2m(\vec{v}\times\vec{\omega})_x =$$

$$= -2m\omega v_z \sin\theta .$$

Neglecting atmospheric friction,

$$v_z = -gt \quad \Rightarrow \quad t = \sqrt{\frac{2z}{g}} .$$

Then $\ddot{x} = 2\omega g t \sin\theta$, yielding

$$x = \frac{\omega g}{3} \sin\theta \underbrace{t^3}_{\left(\frac{2z}{g}\right)^{3/2}} \quad \left(x(0) = 0, \quad v_x(0) = 0 \right), \text{ or}$$

$$x = \frac{\omega}{3} \sqrt{\frac{(2z)^3}{g}} \sin\theta .$$

If $\theta = \frac{\pi}{2}$ (equator) & $z = 10^2$ m,

$$\boxed{x \simeq 2.2 \text{ cm}}$$

Rigid body equations of motion

In general, a rigid body undergoes $\underbrace{\text{displacement}}_{\substack{\text{3 coords}\\ \text{to describe}\\ \text{translation}}} + \underbrace{\text{rotation}}_{\substack{\text{3 angles}\\ \varphi,\theta,\psi}}$.

If the origin of the body frame is at the center of mass (CoM), we have:

$$\vec{\mathcal{J}} = \underbrace{\vec{R}}_{\substack{\text{radius}\\ \text{vector to}\\ \text{CoM}}} \times \overset{\text{total mass}}{M}\underbrace{\vec{v}}_{\frac{d\vec{R}}{dt}} + \underbrace{\sum_i \vec{r}_i^{\,\prime} \times \vec{p}_i^{\,\prime}}_{\text{wrt CoM}}$$

$$T = \underbrace{\frac{1}{2}Mv^2}_{\substack{\text{kinetic}\\ \text{en. of CoM}}} + \underbrace{\frac{1}{2}\sum_i m_i v_i'^2}_{\text{wrt CoM}}$$

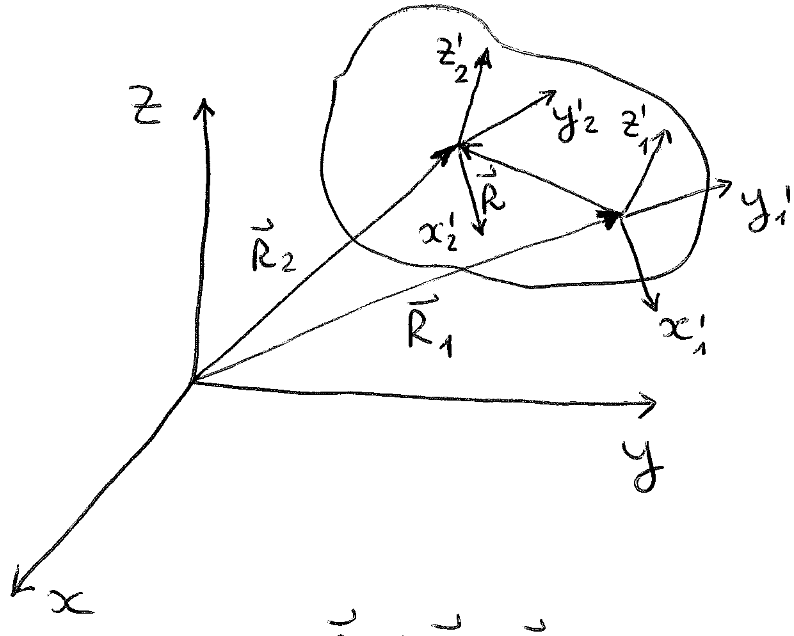
For the rigid body, $\frac{1}{2}\sum_i m_i v_i'^2 \Rightarrow T'(\varphi,\theta,\psi)$

In practice, V can be similarly divided $\Rightarrow$
$\Rightarrow \mathcal{L} = T - V$ is divided as well, and the transl'l & rot'l problems can be solved independently.

Let us focus on rotations:

note that $\underbrace{\vec{\omega}(t)}_{\text{instant. angular velocity}}$ is indep. of the choice of origin of the body frame

Indeed, consider 2 body frames:



$$\vec{R}_1 + \vec{R} = \vec{R}_2$$

Then $$\left(\frac{d\vec{R}_2}{dt}\right)_{lab} = \left(\frac{d\vec{R}_1}{dt}\right)_{lab} + \left(\frac{d\vec{R}}{dt}\right)_{lab} =$$

Using $\left(\frac{d}{dt}\right)_{lab} = \left(\frac{d}{dt}\right)_{body} + \vec{\omega}\times$, we obtain:

$$= \left(\frac{d\vec{R}_1}{dt}\right)_{lab} + \vec{\omega}_1 \times \vec{R} \qquad (*)$$

$\left(\frac{d\vec{R}}{dt}\right)_{body} = 0$ for a rigid body

Alternatively,

$$\left(\frac{d\vec{R}_1}{dt}\right)_{lab} = \left(\frac{d\vec{R}_2}{dt}\right)_{lab} - \left(\frac{d\vec{R}}{dt}\right)_{lab} = \left(\frac{d\vec{R}_2}{dt}\right)_{lab} - \vec{\omega}_2 \times \vec{R} \qquad (**)$$

Now, (*) & (**) give $\underbrace{(\vec{\omega}_1 - \vec{\omega}_2)}_{\text{must be collinear with } \vec{R}} \times \vec{R} = 0$

But $\vec{R}$ is arbitrary, so the only solution is $\underline{\underline{\vec{\omega}_1 = \vec{\omega}_2}}$.

Now, for a rigid body rotation,

$$\vec{J} = \sum_i m_i (\vec{r}_i \times \vec{v}_i)$$

$\vec{r}_i$ is fixed in the body frame:

$$\boxed{\vec{v}_i = \vec{\omega} \times \vec{r}_i}\text{, yielding}$$

$$\vec{J} = \sum_i m_i [\vec{r}_i \times (\vec{\omega} \times \vec{r}_i)] =$$

$$(\vec{r} \times (\vec{\omega} \times \vec{r}))_j = e_{jkl} r_k (\vec{\omega} \times \vec{r})_l =$$

$$= e_{jkl} e_{lmn} r_k \omega_m r_n = (\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) \times$$

$$\times\, r_k r_n \omega_m = r_k^2 \omega_j - \omega_k r_k r_j, \text{ or}$$

$$\vec{r} \times (\vec{\omega} \times \vec{r}) = \underbrace{\vec{r}^2}_{= r^2} \vec{\omega} - \vec{r}(\vec{\omega} \cdot \vec{r})$$

$$= \underline{\underline{\sum_i m_i [\vec{\omega} \vec{r}_i^2 - \vec{r}_i (\vec{r}_i \cdot \vec{\omega})]}}.$$

In components,

$$J_x = \omega_x \sum_i m_i (\vec{r}_i^2 - x_i^2) - \omega_y \sum_i m_i x_i y_i - \omega_z \sum_i m_i x_i z_i, \text{ and}$$

similarly for J_y & J_z.

So, $\vec{\mathcal{L}}$ is related to $\vec{\omega}$ through a linear transformation: $\vec{\mathcal{L}} = \underset{\text{3x3 matrix}}{I}\,\vec{\omega}$, or moment of inertia tensor

$$\mathcal{L}_x = I_{xx}\,\omega_x + I_{xy}\,\omega_y + I_{xz}\,\omega_z$$

and similarly for $\mathcal{L}_y$ & $\mathcal{L}_z$.

Here, $I_{xx} = \sum_i m_i (\vec{r}_i^{\,2} - x_i^2)$,

$I_{xy} = -\sum_i m_i x_i y_i$, etc.

For continuous bodies,

$$I_{xx} = \int_V dV\, \rho(\vec{r})\,(r^2 - x^2)\,, \quad \text{etc.}$$

If we rename $x \to x_1$, $y \to x_2$, $z \to x_3$, we have:

$$I_{jk} = \int_V dV\, \rho(\vec{r})\,(r^2 \delta_{jk} - x_j x_k)\,,$$

and similar for discrete systems

If the coords are in fact measured wrt the body frame, I is indep. of time & characterizes the rigid body.

Properties of tensors

In Cartesian 3D space, a tensor T of the N^{th} rank is a quantity with 3^N components $T_{i_1 \dots i_N}$ which transforms as follows under an orthogonal coord. transf'n:

$$T'_{i_1 \dots i_N} = \underbrace{a_{i_1 j_1} a_{i_2 j_2} \dots a_{i_N j_N}}_{\text{elements of } A} T_{j_1 \dots j_N}$$

0^{th} rank $\Rightarrow$ scalar, inv under A

1^{st} rank $\Rightarrow$ vector, $T'_i = a_{ij} T_j$

2^{nd} rank $\Rightarrow$ $T'_{ij} = a_{ik} a_{jl} T_{kl}$

This is just the previously discussed similarity transform'n:

$$T' = A T A^{-1} = A T \tilde{A} .$$

Indeed, $T'_{ij} = a_{ik} T_{kl} \underbrace{a'_{lj}}_{= a_{jl}} = a_{ik} a_{jl} T_{kl}$, just as above

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Two vectors $\vec{A}$ & $\vec{B}$ can be used to construct a 2^{nd} rank tensor:

$$T_{ij} = A_i B_j$$

Indeed,

$$T'_{ij} = A'_i B'_j = a_{ik} A_k \, a_{jl} B_l =$$

$$= a_{ik} a_{jl} \underbrace{A_k B_l}_{T_{kl}} \text{, as expected.}$$

Finally, $\underset{\text{vector}}{\vec{D}} = \overbrace{\underset{\text{tensor}}{T} \cdot \underset{\text{vector}}{\vec{C}}}^{\text{dot product}}$ is defined by

$$D_i = T_{ij} C_j ,$$

and $\underset{\text{vector}}{\vec{E}} = \underset{\text{vector}}{\vec{F}} \cdot T$ is defined by

$$E_i = F_j T_{ji}$$

Further,

$$\underset{\text{scalar}}{S} = \vec{F} \cdot T \cdot \vec{C} = \underbrace{F_i T_{ij} C_j}_{\text{contraction on } i \,\&\, j}$$

Indeed, $S' = (\vec{F} \underbrace{\tilde{A})(A}_{\mathbb{I}} T \underbrace{\tilde{A})(A}_{\mathbb{I}} \vec{C}) = \vec{F} \cdot T \cdot \vec{C} = S$, as expected

If $T_{ij} = A_i B_j$,

$$(T \cdot \vec{C})_i = T_{ij} C_j = A_i B_j C_j \text{, or}$$

$$T \cdot \vec{C} = (\vec{B} \cdot \vec{C}) \vec{A} .$$

Similarly, $(\vec{F} \cdot T)_i = F_j T_{ji} = F_j A_j B_i$, or

$$\vec{F} \cdot T = (\vec{F} \cdot \vec{A}) \vec{B} .$$

The inertia tensor

Consider

$$T = \frac{1}{2}\sum_i m_i v_i^2 \underset{\uparrow}{=} \frac{1}{2}\sum_i m_i \vec{v}_i \cdot (\vec{\omega}\times\vec{r}_i) \;\ominus$$

rigid body

$$\ominus\; \frac{\vec{\omega}}{2}\cdot \underbrace{\sum_i m_i(\vec{r}_i \times \vec{v}_i)}_{\vec{\mathcal{J}}} = \frac{\vec{\omega}\cdot\vec{\mathcal{J}}}{2} = \frac{\vec{\omega}\cdot \mathrm{I}\cdot\vec{\omega}}{2}.$$

If $\vec{\omega} = \omega\vec{n}$,

$$T = \frac{\omega^2}{2}\,\vec{n}\cdot\mathrm{I}\cdot\vec{n} \equiv \frac{1}{2}\tilde{I}\omega^2, \quad \text{where}$$

$$\tilde{I} = \vec{n}\cdot\underset{\Uparrow}{\mathrm{I}}\cdot\vec{n} = \sum_i m_i\left[r_i^2 - (\vec{r}_i\cdot\vec{n})^2\right] \;\Leftarrow$$ moment of inertia about rot'n axis

$$I_{jk} = \sum_i m_i\left(r_i^2\delta_{jk} - x_{i,j}\,x_{i,k}\right) \quad \text{gives}$$

$$n_j I_{jk} n_k = \sum_i m_i\Big(r_i^2 \underbrace{n_j^2}_{1} - (n_j x_{i,j})(n_k x_{i,k})\Big).$$

Furthermore,

$$(\vec{r}\times\vec{n})\cdot(\vec{r}\times\vec{n}) = e_{jkl}\, r_k n_l\, e_{jmn}\, r_m n_n =$$

$$= \left[\delta_{km}\delta_{ln} - \delta_{kn}\delta_{lm}\right] r_k r_m n_l n_n =$$

$$= r_k^2 \underbrace{n_l^2}_{1} - (r_k n_k)(r_m n_m) = r^2 - (\vec{r}\cdot\vec{n})^2.$$

Thus,

$$\tilde{I} = \sum_i m_i (\vec{r}_i \times \vec{n})\cdot(\vec{r}_i \times \vec{n}) \overset{\vec{\omega} = \omega \vec{n}}{=}$$

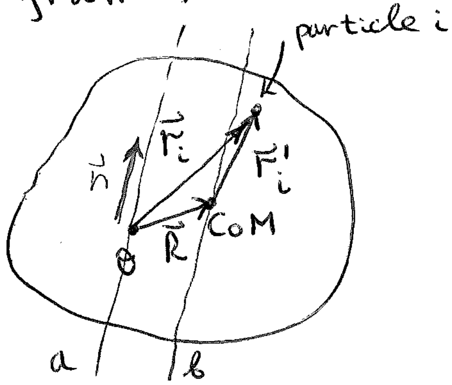
$$= \frac{1}{\omega^2} \sum_i m_i \underbrace{(\vec{\omega}\times\vec{r}_i)}_{\vec{v}_i \text{ in lab frame}} \cdot \underbrace{(\vec{\omega}\times\vec{r}_i)}_{\vec{v}_i} .$$

So, $$\tilde{I} = \frac{1}{\omega^2}\sum_i m_i \vec{v}_i^{\,2} = \frac{2T}{\omega^2} .$$

Note that $\tilde{I}$ depends on $\vec{n}$. Since $\vec{\omega} = \vec{\omega}(t)$ & $\vec{n} = \vec{n}(t)$ in general, $\tilde{I} = \tilde{I}(t)$ as well. However, if $\vec{n} = \text{const}(t)$ [rot'n about a fixed axis], $\tilde{I} = \text{const}(t)$ as well.

Moreover, $\tilde{I}$ depends on the origin of the body frame:

a ∥ b



$\vec{r}_i = \vec{R} + \vec{r}_i'$, yielding

$$\tilde{I} = \sum_i m_i \left[(\vec{r}_i' + \vec{R})\times\vec{n}\right]^2 =$$

$$= \underbrace{M}_{\sum_i m_i} (\vec{R}\times\vec{n})^2 + \sum_i m_i (\vec{r}_i'\times\vec{n})^2 + \underbrace{2\sum_i m_i (\vec{R}\times\vec{n})\cdot(\vec{r}_i'\times\vec{n})}_{-2(\vec{R}\times\vec{n})\cdot(\vec{n}\times\sum_i m_i \vec{r}_i')} .$$

$$\underbrace{\sum_i m_i \vec{r}_i'}_{=0 \text{ for CoM as the origin}}$$

So, $\underbrace{\tilde{I}_a}_{\substack{\text{new}\\\text{origin}}} = \underbrace{\tilde{I}_b}_{\substack{\text{CoM}\\\text{origin}}} + M(\vec{R}\times\vec{n})^2 = \tilde{I}_b + MR^2\sin^2\theta$

θ: angle between $\vec{R}$ & $\vec{n}$

To summarize:

$$T_{rot} = \frac{1}{2}\sum_i m_i(\vec{\omega}\times\vec{r}_i)^2 = \frac{1}{2}\omega_\alpha\omega_\beta\sum_i m_i \times$$
$$\times\left[\delta_{\alpha\beta} r_i^2 - r_{i,\alpha} r_{i,\beta}\right] = \frac{1}{2} I_{\alpha\beta}\,\omega_\alpha\omega_\beta, \quad \text{where}$$

$$I_{\alpha\beta} = \sum_i m_i\left[\delta_{\alpha\beta} r_i^2 - r_{i,\alpha} r_{i,\beta}\right]$$

$\uparrow$ moment of inertia

In the continuous case,

$$I_{\alpha\beta} = \int_V dV\,\rho(\vec{r})\left[\delta_{\alpha\beta} r^2 - r_\alpha r_\beta\right].$$

Note that

$I_{\alpha\beta} = I_{\beta\alpha} \Rightarrow$ 6 indep. components