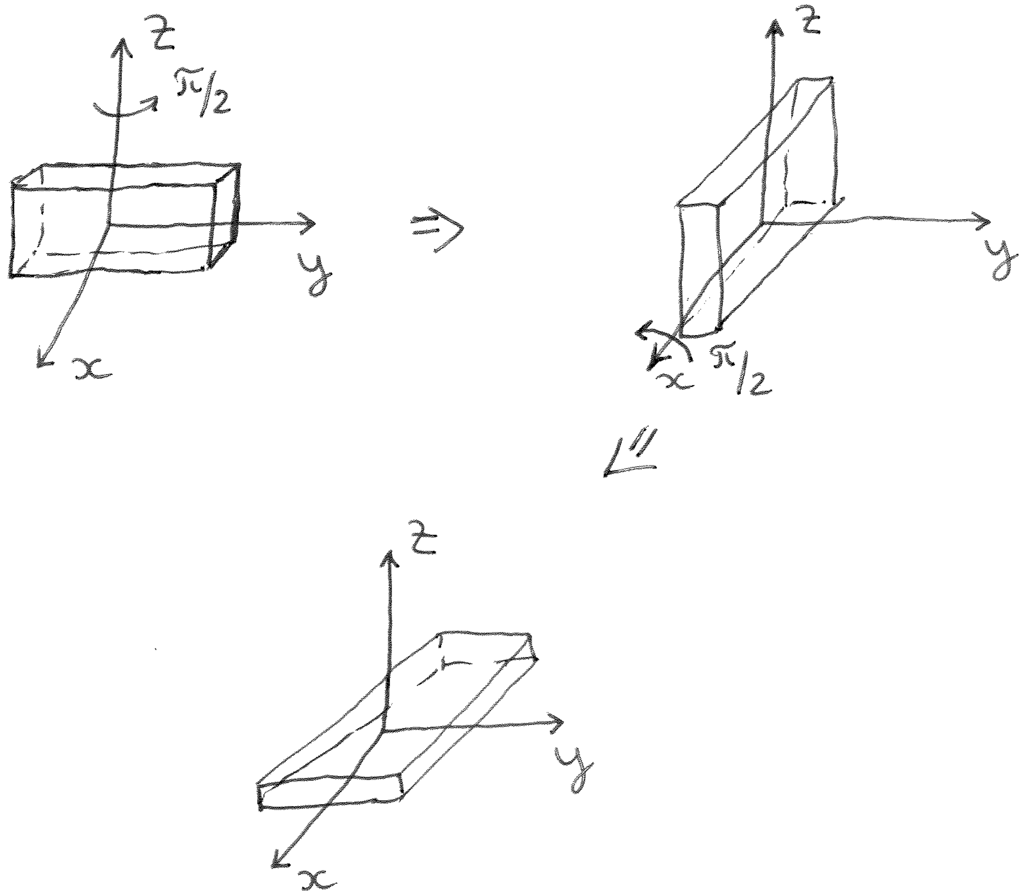
Lecture 11

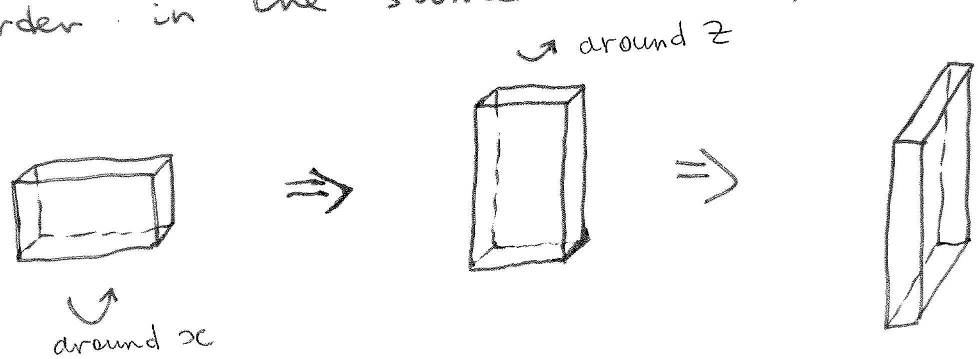
Finite & infinitesimal rotations

Finite rotations do not commute $(AB \neq BA)$:

for example,



If we do these rotations in a different order in the same coord system, we obtain:



The result is not the same in general.

Infinitesimal rotations commute to $\mathcal{O}(\varepsilon^2)$:

$$x'_i = x_i + \varepsilon_{ij} x_j = (\delta_{ij} + \varepsilon_{ij}) x_j \quad , \text{ or}$$

$$\tilde{x}' = \underbrace{(\mathbb{1} + \varepsilon)}_{A} \tilde{x} \, .$$

Now, $A_1 A_2 = (\mathbb{1} + \varepsilon_1)(\mathbb{1} + \varepsilon_2) \approx$ ← $\mathcal{O}(\varepsilon^2)$

$$\approx \mathbb{1} + \varepsilon_1 + \varepsilon_2 = A_2 A_1 \, .$$

Furthermore, $A^{-1} = \mathbb{1} - \varepsilon$ since

$$A A^{-1} = A^{-1} A = (\mathbb{1} + \varepsilon)(\mathbb{1} - \varepsilon) = \mathbb{1}$$

(↑ to $\mathcal{O}(\varepsilon^2)$)

Finally, $\tilde{A} = \mathbb{1} + \tilde{\varepsilon} = A^{-1} = \mathbb{1} - \varepsilon$, s.t.

$\tilde{\varepsilon} = -\varepsilon \Leftarrow$ infinitesimal rot'n matrix is antisymmetric

3 distinct elements in a 3×3 matrix:

$$\varepsilon = \begin{pmatrix} 0 & d\Omega_3 & -d\Omega_2 \\ -d\Omega_3 & 0 & d\Omega_1 \\ +d\Omega_2 & -d\Omega_1 & 0 \end{pmatrix},$$

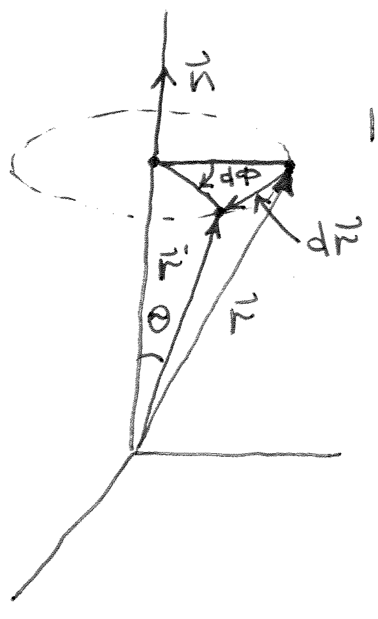
where $d\Omega_j$ are 3 indep. prms specifying the rotation

Next, $\vec{r}' - \vec{r} \equiv d\vec{r}' = \xi \vec{r}$, or

$$\begin{cases} dx_1 = x_2\, d\Omega_3 - x_3\, d\Omega_2, \\ dx_2 = x_3\, d\Omega_1 - x_1\, d\Omega_3, \\ dx_3 = x_1\, d\Omega_2 - x_2\, d\Omega_1 \end{cases} \Rightarrow d\vec{r} = \vec{r} \times d\vec{\Omega}$$

$$\vec{r} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad d\vec{\Omega} = \begin{pmatrix} d\Omega_1 \\ d\Omega_2 \\ d\Omega_3 \end{pmatrix}$$

Now, rotate $\vec{r}$ clockwise by $d\varphi$ through an axis of rotation defined by $\vec{n}$:
(active rot'n)



$|\vec{r}| = |\vec{r}'|$

$dr = r \sin\theta\, d\varphi$

If we define $d\vec{\Omega} = \vec{n}\, d\varphi$,

$$d\vec{r} = \vec{r} \times \vec{n}\, d\varphi = \vec{r} \times d\vec{\Omega}$$

Note however that $d\vec{\Omega}$ is a pseudovector (or an axial vector) rather than a "regular" vector (or a polar vector).

Indeed, define P to be a spatial inversion operator: $P(x) = -x$, $P(y) = -y$, $P(z) = -z$.

Then, clearly, $P(\vec{r}) = -\vec{r} \Rightarrow \vec{r}$ is a polar vector.

Since $\vec{p} = \frac{d\vec{r}}{dt}$, $P(\vec{p}) = -\vec{p} \Rightarrow \vec{p}$ is polar as well.

However, $\vec{L} = \vec{r} \times \vec{p}$ is axial: $P(\vec{L}) = \vec{L}$.
This is the simplest case: a cross product of 2 regular vectors is an axial vector.

Now, from $d\vec{r} = \vec{r} \times \underbrace{d\vec{\Omega}}_{\vec{n}\,d\varphi}$ it is clear that $d\vec{\Omega}$ (and $\vec{n}$) are axial vectors.

What if we wanted to switch the sense of rot'n? (i.e. rotate the body ccw and call this the positive direction)
The main formulas become:

$$d\vec{r} = d\vec{\Omega} \times \vec{r} = (\vec{n} \times \vec{r})\,d\varphi = -(\vec{r} \times \vec{n})\,d\varphi$$

$$\xi = \begin{pmatrix} 0 & -d\Omega_3 & d\Omega_2 \\ d\Omega_3 & 0 & -d\Omega_1 \\ -d\Omega_2 & d\Omega_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{pmatrix} d\varphi .$$

Indeed, $\xi \vec{r} = \begin{pmatrix} -n_3 x_2 + n_2 x_3 \\ n_3 x_1 - n_1 x_3 \\ -n_2 x_1 + n_1 x_2 \end{pmatrix} d\varphi = (\vec{n} \times \vec{r})\,d\varphi$, just as above

$\vec{r} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$\vec{n} = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$ in lab frame

Finally, note that

$$\mathcal{G} = n_1 \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}}_{M_1} d\varphi + n_2 \underbrace{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}}_{M_2} d\varphi +$$

$$+ n_3 \underbrace{\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{M_2} d\varphi = \underbrace{n_i M_i}_{\text{sum implied}} d\varphi$$

$$[M_i, M_j] = M_i M_j - M_j M_i = e_{ijk} M_k$$

Lie algebra of the rotation group (non-abelian)

Vector rotation

How does $\vec{G}$ (a vector or a pseudovector) change under an infinitesimal rot'n?

In time dt, we have:

$$d\vec{G}_{lab} = d\vec{G}_{body} + d\vec{G}_{rot}$$

If $\vec{G}$ is fixed in the body frame, $d\vec{G}_{body} = 0$ and $d\vec{G}_{lab} = d\vec{G}_{rot} = \underbrace{d\vec{\Omega} \times \vec{G}}$

ccw body rot'n is assumed to be positive

In general,

$$\left(\frac{d\vec{G}}{dt}\right)_{lab} = \left(\frac{d\vec{G}}{dt}\right)_{body} + \vec{\omega} \times \vec{G}, \text{ where}$$

$$\vec{\omega} = \frac{d\vec{\Omega}}{dt}$$

Like $d\vec{\Omega}$, $\vec{\omega}$ lies along the instantaneous axis of rotation. $|\vec{\omega}|$ gives the instantaneous rate of rot'n.

Thus $\underbrace{\left(\frac{d}{dt}\right)_{lab} = \left(\frac{d}{dt}\right)_{body} + \vec{\omega}\times}_{\text{operator acting on any vector}}$

Now, consider a general transform:

$$\underbrace{G_i}_{lab} = \tilde{a}_{ij} \underbrace{G_j'}_{body} = a_{ji} G_j' .$$

Then $dG_i = a_{ji} dG_j' + da_{ji} G_j'$.

If lab & body frames coincide @ t,

$$G_i = G_i' \Rightarrow A = \mathbb{I} \Rightarrow a_{ji}\, dG_j' = dG_i'$$

$$\quad \| \tilde{A}$$

We have: $\underbrace{dG_i}_{\text{change of } \vec{G} \text{ in lab frame}} = \underbrace{dG_i'}_{\text{change of } \vec{G} \text{ in body frame}} + \underbrace{da_{ji}}_{\text{infinitesimal rot'n}} G_j'$

Next, $da_{ji} = (\tilde{\xi})_{ij} = -\xi_{ij}$

But $-\xi_{ij} = -e_{ijk}\, d\Omega_k = e_{ikj}\, d\Omega_k$.

Now, $dG_i = dG_i' + e_{ikj}\, d\Omega_k \underbrace{G_j}_{"G_j'}$. G_j', see above

Finally, $dG_i = dG_i' + (d\vec{\Omega} \times \vec{G})_i$, same as the results above.