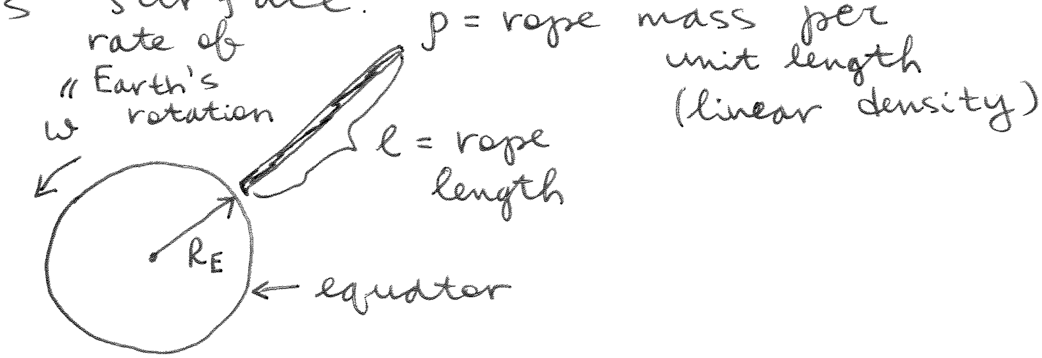
Final

(1.) A "skyhook" satellite concept considers placing a long rope in orbit in a way that the rope appears suspended above a fixed point on the equator. In other words, the bottom of the rope hangs just above the Earth's surface.



Find the length of the rope, ℓ, and the critical frequency, ω_{crit}, above which the skyhook satellite is impossible to implement. Is $\omega < \omega_{crit}$ on Earth?

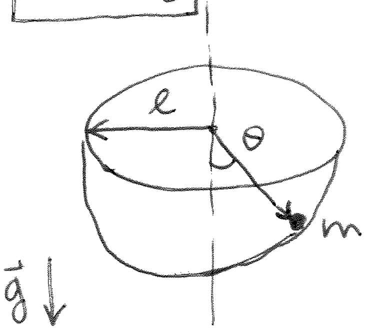
Useful numbers:

Earth's radius

$$\begin{cases} R_E = 6.4\times10^{6}\ \mathrm{m}, \\ \omega = 7.3\times10^{-5}\ \mathrm{s}^{-1}, \\ \dfrac{GM}{R_E^2} = g = 9.8\ \dfrac{\mathrm{m}}{\mathrm{s}^2} \end{cases}$$

(2.) A spherical pendulum is a particle of mass m constrained to move on the surface of the sphere of radius l.

$\boxed{0<\theta<\frac{\pi}{2}}$

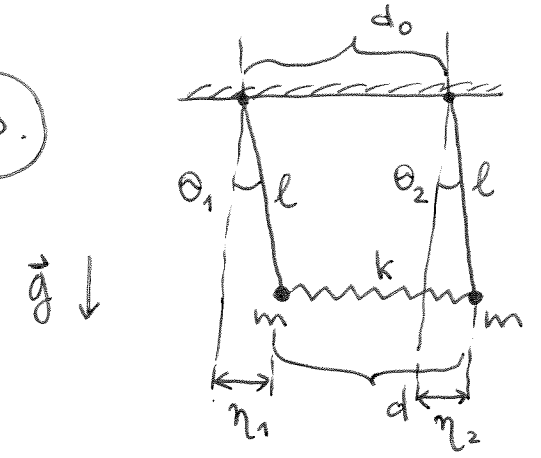


Using θ and φ as generalized coordinates, write down $\mathcal{L}$ and H for this system.

(a) Argue using Hamilton's EoM that a circular orbit with $\theta = \theta_0$ is stable, and find p_φ corresponding to a given value of θ_0.

(b) Expand H around θ_0 and show that the resulting motion corresponds to that of a simple harmonic oscillator with $\omega(\theta_0) = \sqrt{\frac{g}{l\cos\theta_0}(1+3\cos^2\theta_0)}$.

③



Consider two identical pendula moving in a common plane and connected by a spring of length d and force constant k. When both pendula hang vertically, the spring is unstretched and its length $d = d_0$. Consider small oscillations about this equilibrium state. Using the small-angle approximation and employing η_1, η_2 (horizontal displacements of the pendula) as generalized coordinates,

(a) Determine T & V for this system

(b) Write down EoM for $\eta_1(t)$ & $\eta_2(t)$

(c) Find the frequencies of normal modes

(d) Determine $\eta_1(t)$ & $\eta_2(t)$ in the case when the system is at rest at $t=0$ and the left mass is displaced by $+0.01 d_0$ from its vertical position, while the right mass stays vertical at $t=0$. Discuss the nature of the corresponding normal modes.

4. Consider a system described by

$$\begin{cases} T = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2)(q_1^2 + q_2^2) & \text{and} \\ V = \dfrac{1}{q_1^2 + q_2^2} \end{cases}$$

Use the cylindrical coordinates

$$\begin{cases} \rho = \sqrt{q_1^2 + q_2^2}, \\ \varphi = \tan^{-1} \dfrac{q_2}{q_1} \end{cases}$$

(a) Find the Lagrangian $\mathcal{L}$ in terms of ρ, φ and their time derivatives

(b) What are the constants of the motion?

(c) Find the Hamiltonian H for this system

(d) Write down the Hamilton-Jacobi (HJ) equation. Use the HJ approach to obtain an implicit integral equation for $\rho(t)$ [you do not need to solve the equation!] Explain how many constants are present in this equation, as well as the physical meaning of each constant.