

Liouville's theorem

Lecture 26

Suppose that the system is governed by classical mechanics but the number of particles is too great to keep track of all the microscopic trajectories. Then, ^{even} if we knew all the initial conditions, we cannot obtain the solution for later times. The best we can do is to impose some constraints (such as the total $E = \text{const}$) and consider an ensemble of systems satisfying these constraints. Each system is a point in phase space at any given $t \Rightarrow$ the ensemble of systems is a distribution of such points.

Now, consider $D(q, p)$, density of such systems in the neighborhood of (q, p) .
At equilibrium, $D(q, p)$ does not depend on t .

We can formally consider the motion of the systems as that of incompressible "liquid" in phase space, and apply the continuity eq'n to it:

(the total # systems stays const with t)

$$3D: \frac{\partial p}{\partial t} + \vec{\nabla} \cdot (p \vec{v}) = 0 \Rightarrow \vec{\nabla} \cdot (p \vec{v}) = 0 \text{ at steady state}$$

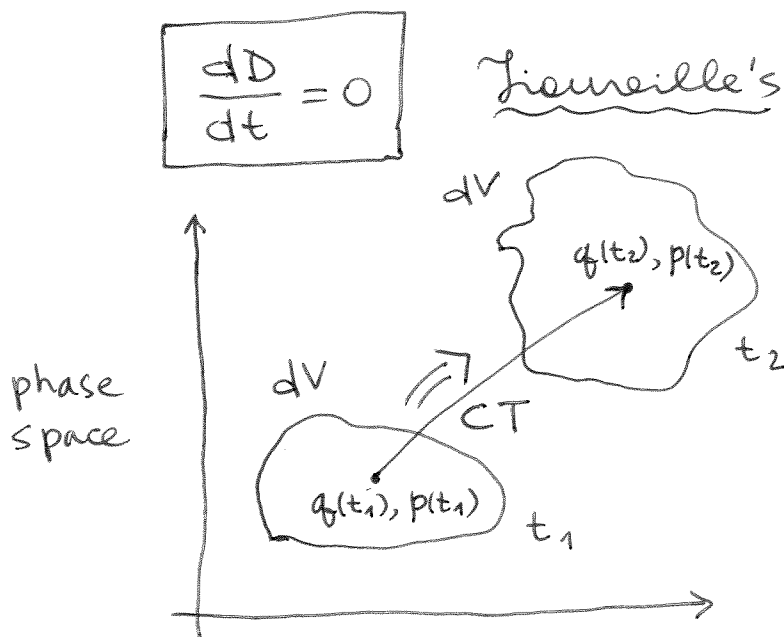
In phase space, this generalizes to

$$\underbrace{\frac{\partial}{\partial q_i} (D \dot{q}_i)}_{\text{implicit sums}} + \underbrace{\frac{\partial}{\partial p_i} (D \dot{p}_i)}_{\text{implicit sums}} = 0, \text{ or}$$

$$\dot{q}_i \frac{\partial D}{\partial q_i} + \dot{p}_i \frac{\partial D}{\partial p_i} + D \left[\underbrace{\frac{\partial \dot{q}_i}{\partial q_i}}_{\frac{\partial^2 H}{\partial q_i \partial p_i}} + \underbrace{\frac{\partial \dot{p}_i}{\partial p_i}}_{-\frac{\partial^2 H}{\partial p_i \partial q_i}} \right] = 0$$

$$\text{Then } \dot{q}_i \frac{\partial D}{\partial q_i} + \dot{p}_i \frac{\partial D}{\partial p_i} = \frac{dD}{dt} = 0$$

$\uparrow$
 $\frac{\partial D}{\partial t} = 0$ at equil.



Shown before
that infinitesimal
volume around
(q, p) does not
change in size
(but in general
changes its shape)

But the number of systems inside the volume also does not change: if a system were to cross the border from the inside it would be infinitesimally close to some system on the border. But then they will have to "travel together" from that point on. So the system cannot cross the border from within (i.e. ~~leave~~ _{the volume}) or from the outside (i.e. come into the volume). But that means that both $dV = \text{const}$ & $\underbrace{dN(q,p)}_{\substack{\text{\# systems} \\ \text{within the volume}}} \text{ is const: } D = \frac{dN}{dV} \text{ is const as well, as shown above}$

Hamilton - Jacobi theory

Idea: as before, simplify Hamiltonian H through a canonical transformation to new variables that are const in time:

$$\begin{cases} q(t) \\ p(t) \end{cases} \Rightarrow \begin{cases} q_0 \\ p_0 \end{cases}$$

↪ ICs at $t=0$

This transformation, if found, can be used to get $\begin{cases} q = q(q_0, p_0, t) \\ p = p(q_0, p_0, t) \end{cases} \Leftarrow$ the solution of the mechanical problem

Recall that

$$K = H + \frac{\partial F}{\partial t}, \text{ and require}$$

$$K = 0 \text{ ("as simple as possible").}$$

$$\text{Then } \begin{cases} \frac{\partial K}{\partial P_i} = \dot{Q}_i = 0 \\ \frac{\partial K}{\partial Q_i} = -\dot{P}_i = 0 \end{cases} \Rightarrow \begin{cases} Q_i = \text{const}, \forall i \\ P_i = \text{const}, \text{ as desired} \end{cases}$$

$$\text{So, } H(q, p, t) + \frac{\partial F}{\partial t} = 0.$$

$$\text{Take } F = F_2(q, P, t), \text{ then } \overset{\text{old momenta}}{p_i} = \frac{\partial F_2}{\partial q_i}$$

$$\text{and } H(q_1, \dots, q_n; \frac{\partial F_2}{\partial q_1}, \dots, \frac{\partial F_2}{\partial q_n}; t) + \frac{\partial F_2}{\partial t} = 0.$$

↗ Hamilton-Jacobi (HJ) eq'n
[partial differential eq'n of $(q_1, \dots, q_n; t)$]
n+1 vars

F_2 is typically called S , Hamilton's principal function. Suppose we have

$$S = S(q_1, \dots, q_n; \underbrace{\alpha_1, \dots, \alpha_{n+1}}_{n+1 \text{ const of integration, expected b/c HJ eq'n only involves partial derivatives of } S \text{ and b/c, physically we want to relate } \alpha_i \text{ s \& } P_i \text{ s, the new const momenta}}; t)$$

Note that if S is a solution, $S + \alpha$ is also a solution $\Rightarrow$ say $\alpha = \alpha_{n+1}$.

Discarding α as uninteresting, we're left with

$$S = S(q_1, \dots, q_n; \underbrace{\alpha_1, \dots, \alpha_n}_{\text{can take } P_i = \alpha_i, \forall i}; t)$$

Then $p_i = \frac{\partial S(q, \alpha, t)}{\partial q_i} \Rightarrow$ can obtain $\underbrace{\alpha_i = \alpha_i(p, q, t)}_{\text{new momenta}}$

Next, $Q_i = \frac{\partial S(q, \alpha, t)}{\partial \alpha_i} \Rightarrow$ can obtain $\beta_i = \beta_i(p, q, t)$
 $\underbrace{Q_i}_{\beta_i = \text{const}} \quad \text{or} \quad \beta_i = \beta_i(\alpha, q, t)$
 can invert this to obtain

$$q_j = q_j(\alpha, \beta, t)$$

Finally, we have $p_j = p_j(\alpha, \beta, t)$

complete solution of EoM

Note that α 's & β 's are not automatically $p_{0,i}$'s & $q_{0,i}$'s, but they can be related to IC's.

Now, examine

$$\frac{dS}{dt} = \underbrace{\frac{\partial S}{\partial q_i}}_{\substack{\uparrow \\ p_i = \text{const}, \forall i}} \dot{q}_i + \underbrace{\frac{\partial S}{\partial t}}_{-H} = p_i \dot{q}_i - H = \mathcal{L}, \text{ or}$$

$$S = \underbrace{\int dt \mathcal{L}}_{\text{indefinite } S} + \text{const}$$

$$\text{If } \frac{\partial H}{\partial t} = 0, \quad H = E \quad \underbrace{\text{conserved total energy}}$$

$$\text{Then } S = \int dt (p_i \dot{q}_i - H) = \underbrace{\int dt p_i \dot{q}_i}_{\text{Hamilton's characteristic f'n}} - Et$$

$\{p_i = p_i(t), q_i = q_i(t)\}$
can be viewed as
 $p_i = p_i(q)$ and
then $\int dq_i p_i(q)$ is
just a f'n of q_i 's
(and $\mathcal{L}$'s)

note that explicit
t-dependence $\Rightarrow$
is gone from W
since

$$\int dq_i p_i \equiv W(q, \mathcal{L})$$

Hamilton's characteristic
f'n

$$\text{So, } S'(q, \mathcal{L}, t) = W(q, \mathcal{L}) - Et, \text{ where}$$

$$\frac{dW}{dt} = \underbrace{\frac{\partial W}{\partial q_i}}_{p_i} \dot{q}_i = p_i \dot{q}_i, \text{ consistent with above}$$

Harmonic oscillator

(1D)

Consider

$$H = \frac{1}{2m} (p^2 + m^2 \omega^2 q^2) = E$$

$n=1$

$$\omega = \sqrt{\frac{k}{m}}$$

↙ HJ eq'n

$$p = \frac{\partial S}{\partial q} \Rightarrow \frac{1}{2m} \left[\left(\frac{\partial S}{\partial q} \right)^2 + m^2 \omega^2 q^2 \right] + \frac{\partial S}{\partial t} = 0$$

↙ here Hamiltonian, K

$n=1$: expect one $\alpha (=E)$ & one

Use $S = W - Et$: β
 ↙ here momentum ↘ here coord.

$$\frac{1}{2m} \left[\left(\frac{\partial W}{\partial q} \right)^2 + m^2 \omega^2 q^2 \right] = E, \text{ or}$$

↙ const of integration

$$S = \sqrt{2mE} \int dq \sqrt{1 - \frac{m\omega^2 q^2}{2E}} - Et$$

$$\begin{aligned} \frac{\partial W}{\partial q} &= \sqrt{2mE - m^2 \omega^2 q^2} = \\ &= \sqrt{2mE} \sqrt{1 - \frac{m\omega^2 q^2}{2E}} \end{aligned}$$

$$\begin{aligned} \text{Now, } \beta &= \frac{\partial S}{\partial E} = \int dq \frac{1}{2 \sqrt{2mE - m^2 \omega^2 q^2}} 2m - t = \\ &= \sqrt{\frac{m}{2E}} \int dq \frac{2m}{\sqrt{\frac{2m}{E}} \sqrt{2mE - m^2 \omega^2 q^2}} - t = \\ &= \sqrt{\frac{m}{2E}} \int dq \frac{1}{\sqrt{1 - \frac{m\omega^2 q^2}{2E}}} - t, \text{ or} \end{aligned}$$

$$t + \beta = \frac{1}{\omega} \sin^{-1} \left[q \sqrt{\frac{m\omega^2}{2E}} \right]. \quad (*)$$

Insert (*):

$$q = \sqrt{\frac{2E}{m\omega^2}} \sin(\omega t + \underbrace{\omega\beta}_{\beta' = \text{const}, \text{osc. phase}})$$

————— 0 —————

$$\text{Next, } \underbrace{p}_{\substack{\text{old} \\ \text{momentum}}} = \frac{\partial S}{\partial q} = \frac{\partial W}{\partial q} = \sqrt{2mE - m^2\omega^2 q^2}$$

Substitute q :

$$p = \sqrt{2mE - m^2\omega^2 \frac{2E}{m\omega^2} \sin^2(\omega t + \beta')} =$$

$$= \sqrt{2mE} \cos(\omega t + \beta') = m \dot{q}, \text{ as expected.}$$

How are E & β' related to $p(0) = p_0$ & $q(0) = q_0$?

$$p_0^2 + m^2\omega^2 q_0^2 = 2mE, \text{ energy conservation}$$

Finally, at $t=0$

$$\tan \beta' = m\omega \frac{q_0}{p_0}$$

Thus, $\begin{cases} q_0 = 0 \\ p_0 \neq 0 \end{cases} \Rightarrow \beta' = 0$ osc. starts at its equil. position at $t=0$

So, the new coords are (i) total E ,
(ii) initial phase of the oscillator

Lastly,

$$\begin{aligned} S &= 2E \int dt \cos^2(\omega t + \beta') - Et = \\ &= \int dt \underbrace{2E \left(\cos^2(\omega t + \beta') - \frac{1}{2} \right)}_{\mathcal{L}} \end{aligned}$$

$$\text{Indeed, } \mathcal{L} = \frac{1}{2m} (p^2 - m^2 \omega^2 q^2) =$$

$$= E (\cos^2(\omega t + \beta') - \sin^2(\omega t + \beta')) =$$

$$= 2E \left(\cos^2(\omega t + \beta') - \frac{1}{2} \right), \text{ just as above}$$