Elements of group theory

{ Lecture 25

A group G is a set of objects (called group elements) on which group multiplication is defined. Groups have following properties:

1. Closure: if $a \in G$ & $b \in G$, $ab = c \in G$ as well.
2. Associativity of group multiplication: if $a, b, c \in G$, $a(bc) = (ab)c$.
3. $\exists$ an identity element $I \in G$ s.t. for any $a \in G$, $a = aI = Ia$.
4. Each $a \in G$ has an inverse, $a^{-1} \in G$ s.t. $aa^{-1} = a^{-1}a = I$.

A group is abelian if $ab = ba$ for $\forall\, a, b \in G$.

Ex. $G = \{\underbrace{1, -1, i, -i}_{h=4}\}$ (h = group order)

Multiplication table:

	1	-1	i	$-i$
1	1	-1	i	$-i$
-1	-1	1	$-i$	i
i	i	$-i$	-1	1
$-i$	$-i$	i	1	-1

$\Leftarrow$ can check all 4 group properties using this table

Clearly, the group is abelian (group multiplication is ordinary multiplication).

Moreover, all group elements can be generated from i:

$$i^2 = -1, \quad i^3 = -i, \quad i^4 = 1.$$

($-i$ also works)

$(-i)^2 = -1 \quad (-i)^3 = i, \quad (-i)^4 = 1$ $h=4$

This group is then called C_4, the cyclic group w/4 elements.

cyclic

In general, C_n has the property that $A^n = I$, where A is the generating element

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A subgroup is a set of elements $\in G$ that by themselves form a group. For example, $\{-1, 1\} \in C_4$ is a subgroup.

Two elements of a group are conjugate if

$$aba^{-1} = c, \qquad a, b, c \in G.$$

All group elements c generated by applying all elements $a \in G$ to b are called a class.

It can be argued that a group is divided into disjoint classes, with each element belonging to only one class. For abelian groups, $aba^{-1} = b$, $\forall a \in G \Rightarrow$ each element is its own class

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For ex., consider the dihedral group

$$D_3 = \{I, A, B, C, D, F\}$$

identity element

In general, a dihedral group D_n has $h = 2n$ elements and 2 generators A, F : $\begin{cases} A^n = I, \\ F^2 = I. \end{cases}$

For D_3, the generator A is such that

$A^3 = I$ and $AA = A^2 = B$.

Clearly, $AB = BA = I \Rightarrow B = A^{-1}$, $B^{-1} = A$.

The other generator, F, obeys $F^2 = I$.

Multiplication table:

	I	A	B	C	D	F
I	I	A	B	C	D	F
A	A	B	I	F	C	D
B	B	I	A	D	F	C
C	C	D	F	I	A	B
D	D	F	C	B	I	A
F	F	C	D	A	B	I

Non-abelian group:
e.g.
$$\underbrace{AC}_{F} \neq \underbrace{CA}_{D}$$

4 subgroups: $\{I, C\}$, $\{I, D\}$, $\{I, F\}$

indeed,
$F^2 = C^2 = D^2 = I$,
which implies
$C^{-1} = C$, $D^{-1} = D$, $F^{-1} = F$

3 classes: $\{I\} \Leftarrow aIa^{-1} = I, \ \forall a \in D_3$

$\{A, B\} \Leftarrow A^{-1} = B$, $B^{-1} = A$ give

$$\begin{cases} IAI^{-1} = A, & CA\underbrace{C^{-1}}_{C} = CF = B, \\ A\underbrace{AA^{-1}} = A, & DA\underbrace{D^{-1}}_{D} = DC = B, \\ \underbrace{B}_{A^{-1}}A\underbrace{B^{-1}}_{A} = A, & FA\underbrace{F^{-1}}_{F} = FD = B. \end{cases}$$

It can be checked that aBa^{-1}, $\forall a \in D_3$ also gives $\{A, B\}$.

Finally, the 3^{rd} class is $\{C, D, F\}$.

Group representations

Irreducible representation (IR) T_i: a set of $m \times m$ matrices that obey the group multiplication table

Irreducible: cannot be simultaneously decomposed into lower-order (smaller) matrices

It can be shown that $\boxed{\#\,\text{IRs} = \#\,\text{classes}}$

$$(+) \quad \sum_{i=1}^{k} l_i^2 = \underbrace{h}_{\#\,\text{elements}} \qquad k \leftarrow \#\,\text{IRs}$$

$\Leftarrow l_i$ is the dim'n of i^{th} representation

For ex., for D_3 $\underbrace{k=3}_{\#\,\text{classes}}$ and $h=6$, so that

$$(+) \Rightarrow l_1^2 + l_2^2 + l_3^2 = 6 .$$

This can only be satisfied by

$l_1 = 1$, $l_2 = 1$, $l_3 = 2$ (or permutations)

We therefore have T_1 $(l_1=1)$: each element maps into $+1$ (trivial)

T_2 $(l_2=1)$: $\{1, -1\}$ s.t. $\{I, A, B\} \to 1$

$\{C, D, F\} \to -1$

For ex., $\underbrace{BC}_{D} \to 1 \times (-1) = \underbrace{-1}_{D}$

T_3: $(l_3 = 2)$

[m=2] matrices

Define $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

σ_i's are Pauli matrices

Then
$$\begin{cases} I \to I \\ A \to -\frac{1}{2}(I - i\sigma_2\sqrt{3}) \\ B \to -\frac{1}{2}(I + i\sigma_2\sqrt{3}) \\ C \to \frac{1}{2}(\sqrt{3}\,\sigma_1 + \sigma_3) \\ D \to -\frac{1}{2}(\sqrt{3}\,\sigma_1 - \sigma_3) \\ F \to \sigma_3 \end{cases}$$

For ex.,

$$AB = \frac{1}{4}(I - i\sigma_2\sqrt{3})(I + i\sigma_2\sqrt{3}) =$$

$$= \frac{1}{4}\left[I + 3\underbrace{\sigma_2^2}_{\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I}\right] = I$$, as expected from the multiplication table

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Γ_1 & Γ_2 are unfaithful (many-to-one) representations, while Γ_3 is faithful (one-to-one).

Lie groups & Lie algebras

Elements of a Lie group define a (continuous) manifold: a continuous geometric object with a concept of nearness.

Examples of manifolds: Euclidean space, unit circle.

For a manifold to be a Lie group, group multiplication must be defined and the 4 group properties must be satisfied.

Lie groups are conveniently characterized by Lie algebras: Lie algebra is a set of N vectors, $\{\tau_i\}$, which serve as the basis in some flat vector space.

These vectors satisfy

$$\underbrace{[\tau_i, \tau_j]}_{\text{Lie brackets}} = \tau_i \tau_j - \tau_j \tau_i = \sum_{k=1}^{N} c_{ij}{}^{k} \tau_k ,$$

where $c_{ij}{}^{k} = - c_{ji}{}^{k}$ are structure constants of the Lie algebra.

They also satisfy Jacobi identity:

$$[\tau_i,[\tau_j,\tau_k]] + [\tau_j,[\tau_k,\tau_i]] + [\tau_k,[\tau_i,\tau_j]] = 0$$

Finally, elements of the Lie group can be generated by:

$$a_m = e^{i \sum_{k=1}^{N} \underbrace{\theta_m{}^{k}}_{\text{element of } \vec{\theta}_m = \begin{pmatrix} \theta_m{}^1 \\ \theta_m{}^2 \\ \vdots \\ \theta_m{}^N \end{pmatrix} \text{ which gives } a_m} \tau_k}$$

We often deal with $N=1$:

$$a_m = e^{i \theta_m \tau}$$ ↑ single Lie algebra vector

Ex. 1. Special orthogonal groups:

$SO(n)$

Group elements are rotations in Euclidean space of dimension n, (faithfully) represented by $n\times n$ $\underbrace{\text{rotation}}_{\text{real-valued}}$ matrices with $\underbrace{\det = 1}_{\text{no reflections}}$.

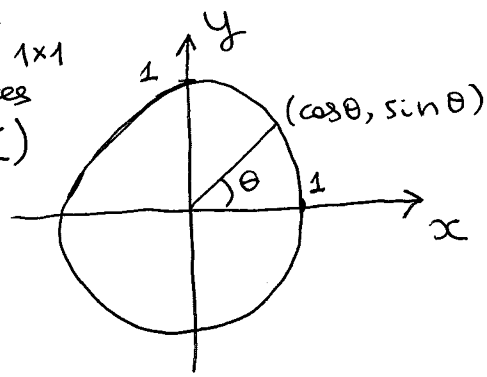
Group multiplication is matrix multiplication.

In particular, $SO(2)$ is the set of all rotations on $\underbrace{R^2}_{n=2 \text{ Euclidean space}}$.

$$\underbrace{SO(2) = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, 0\le\theta<2\pi \right\}}_{\text{an irreducible } \underline{\text{real}} \text{ representation of } SO(2)}$$

This group is isomorphic with

$$\underbrace{U(1)}_{\substack{\text{a group of} \\ \text{complex-valued } 1\times1 \\ \text{unitary matrices} \\ (UU^+ = U^+U = \mathbb{1})}} = \{ e^{i\theta},\ 0\le\theta<2\pi \}$$



Note that both $SO(2)$ and $U(1)$ are abelian.

Clearly, for $U(1)$ $\tau = 1$ and $d(\theta) = e^{i\theta}$.

For $SO(2)$, $\tau = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $i\theta \to \theta$

Indeed, $\tau^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -\mathbb{I}$,

$$\tau^3 = -\tau,$$
$$\tau^4 = -\tau^2 = \mathbb{I},$$
$$\tau^5 = \tau, \text{ etc.}$$

Finally, $d(\theta) = e^{\theta\tau} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \theta\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} -$

$$- \frac{\theta^2}{2!}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{\theta^3}{3!}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \frac{\theta^4}{4!}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \ldots =$$

$$= \begin{pmatrix} 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \ldots & -\theta + \frac{\theta^3}{3!} + \ldots \\ \theta - \frac{\theta^3}{3!} + \ldots & 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \ldots \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

as expected.

2. Special unitary group: $SU(2)$

$SU(n)$ is a group of $n \times n$ unitary matrices with $\det = 1$.
Group multiplication is matrix multiplication.

In particular, $SU(2)$ is a group that can be parameterized using Euler's angles.
For example, for a rotation through θ around unit vector $\vec{n}$, we obtain:

$$a(\theta) = \mathbb{I}\cos\frac{\theta}{2} + i\,\vec{n}\cdot\vec{\sigma}\,\sin\frac{\theta}{2}.$$

For example, if $\vec{n} = \begin{pmatrix}1\\0\\0\end{pmatrix} = \hat{x}$, $\vec{n}\cdot\vec{\sigma} = \sigma_x$

and we have

$$a(\theta) = \mathbb{I}\cos\frac{\theta}{2} + i\underbrace{\sigma_x}_{\begin{pmatrix}0&1\\1&0\end{pmatrix}}\sin\frac{\theta}{2} = \begin{pmatrix}\cos\frac{\theta}{2} & i\sin\frac{\theta}{2}\\ i\sin\frac{\theta}{2} & \cos\frac{\theta}{2}\end{pmatrix}$$

In general, $SU(2)$ elements are

$$a = \begin{pmatrix}\alpha & -\beta\\ \beta^* & \alpha^*\end{pmatrix}, \text{ with } |\alpha|^2+|\beta|^2=1.$$

This is a faithful representation of $SU(2)$.

In this representation, 3D vectors are encoded as

$$V = \begin{pmatrix}V_z & V_x - iV_y\\ V_x+iV_y & -V_z\end{pmatrix}$$

and a rotation is performed as

$$V \to a(\theta)\,V\,a^+(\theta).$$

Indeed, consider $V = \begin{pmatrix}0\\1\\0\end{pmatrix} = \hat{y} \Rightarrow V = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$

Under a rotation around x-axis by θ, we have:

$$V \to \begin{pmatrix} \cos\frac{\theta}{2} & i\sin\frac{\theta}{2} \\ i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} =$$

$$= \begin{pmatrix} -\sin\theta & -i\cos\theta \\ i\cos\theta & \sin\theta \end{pmatrix} \Rightarrow V = \begin{pmatrix} 0 \\ \cos\theta \\ -\sin\theta \end{pmatrix}, \text{ as expected}$$

Finally, with pauli matrices

$$\sigma_i^2 = \mathbb{I} \quad (\forall i) \quad \text{and} \quad \underset{\text{commutator}}{[\sigma_i, \sigma_j]} = \underbrace{2 i e_{ijk}}_{C_{ij}^{k}} \sigma_k .$$

$$\text{anticommutator} \Rightarrow \{\sigma_i, \sigma_j\} = 2\delta_{ij}\mathbb{I}$$

pauli matrices are in fact the vectors in the Lie algebra that corresponds to $SU(2)$:

$$\sigma_x = \sigma_1, \ \sigma_y = \sigma_2, \ \sigma_z = \sigma_3$$

$$d(\theta) = e^{i\frac{\theta}{2}\underbrace{\vec{n}\cdot\vec{\sigma}}_{n_x\sigma_x + n_y\sigma_y + n_z\sigma_z}}$$

So, $n_k \frac{\theta}{2}$ are like θ^k in the Lie group exponentiation formula, and σ_k are like τ_k $(k=1,2,3)$

Indeed,

$$d(\theta) = \mathbb{I} + i\frac{\theta}{2}\vec{n}\cdot\vec{\sigma} - \frac{\theta^2}{2^2 2!}(\vec{n}\cdot\vec{\sigma})^2 - i\left(\frac{\theta}{2}\right)^3 \frac{1}{3!}(\vec{n}\cdot\vec{\sigma})^3 + \dots$$

$$(\vec{n}\cdot\vec{\sigma})^2 = (n_x\sigma_x + n_y\sigma_y + n_z\sigma_z)^2 = \underbrace{\mathbb{I}}_{(n_x^2+n_y^2+n_z^2) = \vec{n}^2 = 1} +$$

$$+ n_x n_y \{\sigma_x, \sigma_y\} + n_x n_z \{\sigma_x, \sigma_z\} +$$

$$+ n_y n_z \{\sigma_y, \sigma_z\} = \mathbb{I}.$$

Then $(\vec{n}\cdot\vec{\sigma})^3 = (\vec{n}\cdot\vec{\sigma})$ and so on.

Finally,

$$d(\theta) = \underbrace{\left(1 - \frac{(\theta/2)^2}{2!} + \dots\right)}_{\cos\frac{\theta}{2}} \mathbb{I} + i\underbrace{\left(\frac{\theta}{2} - \frac{(\theta/2)^3}{3!} + \dots\right)}_{\sin\frac{\theta}{2}}(\vec{n}\cdot\vec{\sigma}) =$$

$$= \mathbb{I}\cos\frac{\theta}{2} + i\,\vec{n}\cdot\vec{\sigma}\,\sin\frac{\theta}{2}$$, as desired.

Finally, note that Pauli matrices satisfy Jacobi's identity:

$$[\sigma_1, \underbrace{[\sigma_2, \sigma_3]}_{2i\sigma_1}] + [\sigma_2, \underbrace{[\sigma_3, \sigma_1]}_{2i\sigma_2}] + [\sigma_3, \underbrace{[\sigma_1, \sigma_2]}_{2i\sigma_3}] = 0,$$

since $[\sigma_i, \sigma_i] = 0, \ \forall i$

Finally, recall that

$$u(\alpha) = e^{\alpha \hat{G}} u\Big|_0 \quad , \text{ where } \quad \hat{G} u = [u, G]$$

In particular,

$$u(t) = e^{t\hat{H}} u\Big|_0 \quad , \text{ where } H \text{ is the Hamiltonian}$$

Now we recognize $u(\alpha)$ as elements of a Lie group. G is a basis vector of the corresponding Lie algebra. For example,
an infinitesimal generator of the Lie group

H is the generator of the Lie group that describes the trajectory of the system in phase space.

Finally, group multiplication is the application of the PB as shown above.

In summary,

$$\underbrace{d^{(\theta)} = e^{i\theta\tau}}_{N=1 \text{ case}} \quad \leftrightarrow \quad u(t) = e^{t\hat{H}} u\Big|_0$$