Solutions

Midterm Exam: E&M II 504

Spring 2018

Problem 1 [20 points]

Consider a particle with charge q and mass m moving in a uniform magnetic field $\vec{B} = B\hat{z}$ which is allowed to change adiabatically with time. Let us assume for simplicity that the initial velocity of the particle, $\vec{v}_\perp(0)$, is perpendicular to $\vec{B}$. The adiabatic condition means that the magnetic field changes negligibly during any one orbit, so that the particle orbit remains nearly circular.

1. Find the orbital angular speed $\vec{\omega}_B$ of the particle's motion. Write the result in vector form (i.e. find both amplitude and direction of $\vec{\omega}_B$).

2. Show that the orbital magnetic moment of the charged particle is given by

$$\vec{\mathfrak{m}} = -\frac{mv_\perp^2}{2}\frac{\vec{B}}{B^2}, \tag{1}$$

where $\vec{v}_\perp(t)$ is the component of the particle velocity perpendicular to $\vec{B}$.

3. Show that the power delivered to the charge by the time-dependent field is

$$P = \frac{mv_\perp^2}{2}\frac{d}{dt}(\ln B). \tag{2}$$

4. Use the result above to show that $\vec{\mathfrak{m}}$ is invariant.

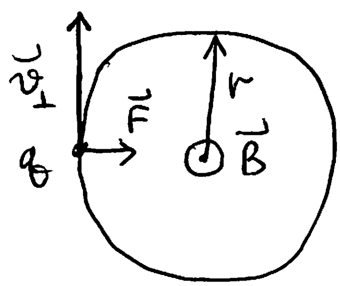
1. The Lorentz force is acting on the charge: $\dfrac{mv_\perp^2}{r} = qv_\perp B$.

r radius of the orbit

Since $\omega_B = \dfrac{v_\perp}{r}$,

$$m\omega_B = qB, \text{ or}$$

$$\omega_B = \frac{qB}{m}.$$



$$\vec F = q\,\vec v_\perp \times \vec B$$

Positive q revolves clockwise around $\vec B$:

$$\vec\omega_B = -\frac{q\vec B}{m}$$

2. $$\vec m = \underbrace{I}_{\frac{q}{T}=\frac{q\omega_B}{2\pi}}\overbrace{\vec A}^{-\pi r^2\hat z} = -\left(\frac{q\omega_B}{2\pi}\right)\pi r^2\hat z = -\left(\frac{q\omega_B}{2\pi}\right)\pi\left(\frac{v_\perp}{\omega_B}\right)^2\hat z =$$

$$= -\frac{mv_\perp^2}{2}\frac{\vec B}{B^2}.$$

3. $$\vec\nabla\times\vec E = -\frac{\partial\vec B}{\partial t} \Rightarrow \oint\vec E\cdot d\vec l = -\frac{d\varphi_B}{dt}, \text{ or}$$

$$E(2\pi r) = -\pi r^2\frac{dB}{dt} \Rightarrow E = -\frac{r}{2}\frac{dB}{dt}.$$

$$\vec E \uparrow\uparrow \vec v_\perp$$

Then the power $P = q\vec E\cdot\vec v_\perp = q|E|v_\perp =$

$$= q\left(\frac{r}{2}\frac{dB}{dt}\right)v_\perp \underset{\uparrow\; r=\frac{v_\perp m}{qB}}{=} \frac{mv_\perp^2}{2}\frac{d}{dt}(\log B).$$

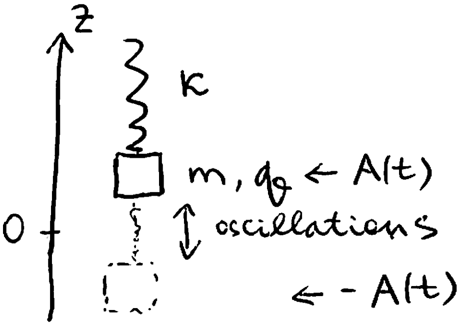
4. Note that here the particle's KE is not conserved: $\dfrac{d}{dt}\left(\dfrac{mv_\perp^2}{2}\right) = \dfrac{mv_\perp^2}{2}\dfrac{d}{dt}(\log B)$,

$$\frac{d}{dt}\left(\log\left(\frac{mv_\perp^2}{2}\right)\right) = \frac{d}{dt}(\log B) \Rightarrow \frac{mv_\perp^2}{2} = B\times\text{constant}.$$

But $$\vec m = -\frac{mv_\perp^2}{2}\frac{\hat z}{B} = \underbrace{-\text{constant}\times\hat z}_{\text{invariant!}}.$$

Problem 2 [10 points]

Consider a simple harmonic oscillator which consists of a particle with charge q and mass m connected to an ideal spring with a spring constant κ (there is no gravity). The particle is set oscillating with the initial amplitude $A(0)$. Assuming non-relativistic motion, find the radiated power per unit solid angle and the total radiated power. Find $E(t)$, where E is the total energy of the harmonic oscillator and t is time.



$m\ddot{z} = -\kappa z$ gives

$$\omega^2 = \frac{\kappa}{m}$$

$$z(t) = A\cos(\omega t)$$

[assume initial phase $= 0$]

The dipole moment $p(t) = q\, z(t) =$

$$= q A \cos(\omega t) = q A \,\mathrm{Re}\{e^{-i\omega t}\}.$$

In Jackson, we used the convention

$$p(\vec{x}, t) = p(\vec{x})\, e^{-i\omega t},$$ (Re part implied)

which leads to $\underline{\underline{p = qA}}$.

$$k = \frac{\omega}{c} = \frac{1}{c}\sqrt{\frac{\kappa}{m}}$$

(9.23)

Then $$\frac{dP}{d\Omega} = \frac{c^2 Z_0}{32\pi^2} k^4 p^2 \sin^2\theta =$$

power per unit solid angle

$$= \frac{Z_0}{32\pi^2 c^2} \frac{\kappa^2}{m^2} q^2 A^2 \sin^2\theta .$$

$$\begin{cases} Z_0 = \sqrt{\dfrac{\mu_0}{\epsilon_0}}, \\ c = \dfrac{1}{\sqrt{\mu_0 \epsilon_0}} \end{cases}$$

(9.24)

$$P = \frac{c^2 Z_0 k^4}{12\pi} q^2 A^2 = \frac{q^2 A^2 Z_0 K^2}{12\pi c^2 m^2} =$$

total power

$$\frac{Z_0}{c^2} = \frac{1}{\epsilon_0 c^3} \qquad = \frac{q^2 A^2 K^2}{12\pi \epsilon_0 c^3 m^2}.$$

As the oscillator is losing energy through radiation, A decreases:

$$\frac{dE}{dt} = -\frac{q^2 A^2 K^2}{12\pi \epsilon_0 c^3 m^2}, \quad \text{where}$$

$$E(t) = \frac{1}{2} K A^2(t).$$

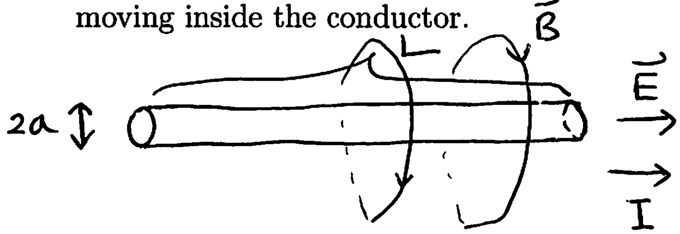
So, $$\frac{dE}{dt} = -\frac{q^2 K}{6\pi \epsilon_0 c^3 m^2} E, \quad \text{or}$$

$$E(t) = E(0)\, e^{-t/\tau}, \quad \text{where}$$

$$\tau = \frac{6\pi \epsilon_0 c^3 m^2}{q^2 K}.$$

Problem 3 [10 points]

A long cylindrical ohmic conductor of radius a and length L carries a current I due to a constant potential difference V between its ends. Calculate the Poynting vector $\vec{S}$ (both its magnitude and direction) and use it to obtain work per unit time done on the charges moving inside the conductor.



$$|\vec{E}| = \frac{V}{L},$$

$$|\vec{B}| = \frac{\mu_0 I}{2\pi a}.$$

$$\vec{S} = \vec{E} \times \vec{H} \Rightarrow |\vec{S}| = \frac{VI}{2\pi a L}.$$

$\vec{S}$ is pointing $\perp$ to the surface of the wire & away from it.

Recall that in a steady-state situation,

$$\underbrace{\int_V \vec{\nabla}\cdot\vec{S}\, d^3x}_{\parallel} = -\underbrace{\int_V \vec{J}\cdot\vec{E}\, d^3x}_{\text{work per unit time}}$$

$$\int_S \vec{S}\cdot\vec{n}\, ds = VI,$$

surface $\int$

which is the total power dissipated into heat in the system