$\boxed{\text{HW \# 7}}$

11.6 a) K - lab frame (ignore Earth motion)
K' - instantaneous rest frame of the ship

u - ship velocity in K
u' - ship velocity in K'
v - velocity of K' wrt K

Parallel velocities (ship is moving along the z-axis):

$$u = \frac{u'+v}{1+\frac{vu'}{c^2}} \qquad (*)$$

inertial frames

$$\text{Then } \frac{du}{dt} = \frac{\left(1+\frac{u'v}{c^2}\right)\left(\frac{du'}{dt}+\cancel{\frac{dv}{dt}}\right) - \frac{u'+v}{c^2}\left(\frac{du'}{dt}v + \cancel{\frac{dv}{dt}}u'\right)}{\left(1+\frac{u'v}{c^2}\right)^2}$$

$$= \frac{1-\frac{v^2}{c^2}}{\left(1+\frac{u'v}{c^2}\right)^2}\,\frac{du'}{dt}.$$

But $u'=0$ instantaneously, and

$$\frac{du'}{dt} = \frac{du'}{dt'}\underbrace{\frac{dt'}{dt}}_{1/\gamma}, \text{ since } t' \text{ is proper time}$$

Finally, $$\frac{du}{dt} = \left(1-\frac{v^2}{c^2}\right)^{3/2}\underbrace{\frac{du'}{dt'}}_{=g}$$

Note that $u=v$ if $u'=0$ $\Leftarrow$ follows from $(*)$

Then $$\int^{u}\frac{du}{\left(1-\frac{u^2}{c^2}\right)^{3/2}} = \int^{t} g\,dt, \text{ or}$$

$$\frac{u}{\sqrt{1-\frac{u^2}{c^2}}} = gt \Rightarrow u = \frac{gt}{\sqrt{1+\left(\frac{gt}{c}\right)^2}}$$

Now,

$$\int^{t'} dt' = \int^{t}\frac{dt}{\gamma} = \int^{t}\sqrt{1-\frac{u^2}{c^2}}\,dt, \text{ or}$$

$$t' = \int^{t} dt\sqrt{1-\frac{g^2t^2}{c^2\left(1+\frac{g^2t^2}{c^2}\right)^2}} =$$

$$= \int^{t} dt\,\frac{1}{\sqrt{1+\left(\frac{gt}{c}\right)^2}} = \frac{c}{g}\sinh^{-1}\left(\frac{gt}{c}\right).$$

Finally, $$t = \frac{c}{g}\sinh\left(\frac{gt'}{c}\right)$$

$t' = 5\text{ yr} \Rightarrow t \simeq 86\text{ yr}$

$g = 9.86\ \frac{m}{s^2}$

$c = 3\times10^8\ \frac{m}{s}$

The total journey is $4t = 344$ yr.

Thus the year on Earth is $\underline{\underline{2444}}$.

b) In the first leg, the rocket ship traveled:

$$L = \int_0^{t=86\,yr \equiv T} u\,dt = \int_0^T \frac{gt}{\sqrt{1+\left(\frac{gt}{c}\right)^2}}\,dt =$$

$$= \frac{c^2}{g}\left[\sqrt{1+\underbrace{\frac{g^2T^2}{c^2}}_{\gg 1}} - 1\right] \simeq cT - \frac{c^2}{g} \simeq$$

$$\simeq 85 \text{ light-years}$$

The total distance in one dir'n is $2L = 170$ lyr.

Note that a beam of light would travel 172 lyr in 172 yr, not much further!

(11.11) $A_1 = e^{L}, \quad A_2 = e^{L+\delta L}$

Consider $\begin{cases} A_1(\lambda) = e^{\lambda L}, \\ A_2(\lambda) = e^{\lambda(L+\delta L)} \end{cases}$

Now, $A(\lambda) = A_2(\lambda) A_1^{-1}(\lambda) \; {}_{*}\!= e^{\lambda(L+\delta L)} e^{-\lambda L}$.

Consider the Taylor series of $A(\lambda)$:

$$A(\lambda) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} A^{(n)}(0)$$

$\underline{n=0}$: $A^{(0)}(0) = 1$

$\underline{n=1}$: $\lambda A^{(1)}(0) = \lambda(L+\delta L) - \lambda L = \lambda \delta L \Rightarrow$

$$\Rightarrow A^{(1)}(0) = \underline{\underline{\delta L}}.$$

Proceed by induction:
assume $A^{(n)}(0) = [L, A^{(n-1)}(0)]$.

Then $\displaystyle \frac{d}{d\lambda} \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} A^{(n)}(0) = \sum_{n=1}^{\infty} \frac{\lambda^{n-1}}{(n-1)!} \underbrace{A^{(n)}(0)}_{[L, A^{(n-1)}(0)]} \;=$

$\uparrow$ $n=0$ term is indep. of λ

$$\overset{m=n-1}{=} \sum_{m=0}^{\infty} \frac{\lambda^m}{m!} A^{(m+1)}(0) \Rightarrow A^{(m+1)}(0) = [L, A^{(m)}(0)]$$

Finally,

$$A(\lambda=1) = 1 + \delta L + \frac{1}{2!}[L, \delta L] + $$

$$+ \frac{1}{3!}[L,[L,\delta L]] + \ldots \, , \text{ as } \underline{\underline{\text{desired}}}.$$

12.1

a) $$L = -\frac{m}{2} u_\alpha u^\alpha - \frac{q}{c} u_\alpha A^\alpha =$$

$$= -\frac{m}{2} g_{\alpha\beta}\, u^\beta u^\alpha - \frac{q}{c} g_{\alpha\beta}\, u^\beta A^\alpha .$$

Then $$\frac{\partial L}{\partial u^\gamma} = -\frac{m}{2} g_{\alpha\beta}\left[\delta^\beta{}_\gamma u^\alpha + u^\beta \delta^\alpha{}_\gamma\right] -$$

$$- \frac{q}{c} g_{\alpha\beta}\, \delta^\beta{}_\gamma A^\alpha = -\frac{m}{2}\left[u_\gamma + u_\gamma\right] - \frac{q}{c} A_\gamma ,$$

$$\frac{\partial L}{\partial x^\gamma} = -\frac{q}{c} g_{\alpha\beta}\, u^\beta \partial_\gamma A^\alpha = -\frac{q}{c} u_\alpha \partial_\gamma A^\alpha$$

EL eq's: $$\frac{d}{d\tau}\frac{\partial L}{\partial u^\alpha} = \partial_\gamma L \quad , \text{or}$$

proper time

$$\frac{d}{d\tau}\left[m u_\gamma + \frac{q}{c} A_\gamma\right] = \frac{q}{c} u^\alpha \partial_\gamma A_\alpha .$$

$$m\frac{du_\gamma}{d\tau} = \frac{q}{c}\Big[u^\alpha \partial_\gamma A_\alpha - \underbrace{\frac{dA_\gamma}{d\tau}}_{\frac{dx^\alpha}{d\tau}\frac{\partial A_\gamma}{\partial x^\alpha} = u^\alpha \partial_\alpha A_\gamma}\Big],$$

or $$m\frac{du_\gamma}{d\tau} = \frac{q}{c} u^\alpha \Big[\underbrace{\partial_\gamma A_\alpha - \partial_\alpha A_\gamma}_{F_{\gamma\alpha}}\Big] =$$

$$= \frac{q}{c} F_{\gamma\alpha} u^\alpha .$$

Covariant Lorentz force eq'n

b) $p^\alpha = -\dfrac{\partial L}{\partial u_\alpha} = m u^\alpha + \dfrac{q}{c} A^\alpha$

canonical momentum

$$H = p^\alpha u_\alpha + L = \frac{m}{2} \underbrace{u^\alpha u_\alpha}_{c^2} =$$

signs consistent with (12.34)

$$= \frac{1}{2m}\left(p^\alpha - \frac{q}{c} A^\alpha\right)\left(p_\alpha - \frac{q}{c} A_\alpha\right)$$

$$H = \frac{mc^2}{2} \leftarrow \text{Lorentz inv}$$

In space-time coordinates,

$$p^\alpha - \frac{q}{c} A^\alpha = \begin{pmatrix} p^0 - \frac{q}{c}\varphi \\ \vec{p} - \frac{q}{c}\vec{A} \end{pmatrix}.$$

Then

$$H = \frac{1}{2m}\left[(p^0)^2 - \vec{p}^{\,2} + \frac{q^2}{c^2}(\varphi^2 - \vec{A}^2) + \frac{2q}{c} \times (\vec{p}\cdot\vec{A} - p^0\varphi) \right].$$