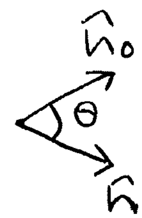
HW #6 solutions

10.9 a) Use the 1st Born appr'n (ϵ_r close to 1):

$$\frac{\vec{\epsilon}^* \cdot \vec{A}_{sc}^{(1)}}{D_0} = \frac{k^2}{4\pi} \int d^3x \, e^{i\vec{q}\vec{x}} \left[\vec{\epsilon}^* \cdot \vec{\epsilon}_0 \frac{\delta\epsilon}{\epsilon_0} + \right.$$

$$\left. + (\hat{n}_0 \times \vec{\epsilon}_0)(\hat{n} \times \vec{\epsilon}^*) \frac{\delta\mu}{\mu_0} \right] \quad (10.31)$$

Here, $\vec{q} = k(\hat{n}_0 - \hat{n})$, so that



$q = 2k\sin\frac{\theta}{2}$

$$q^2 = 2k^2(1-\cos\theta) = 4k^2\sin^2\frac{\theta}{2}.$$

Further, $\delta\mu = 0$ and

$$\frac{\delta\epsilon}{\epsilon_0} = \begin{cases} \epsilon_r - 1 & r \le a \\ 0 & r > a \end{cases}$$

Then
$$\frac{\vec{\epsilon}^* \cdot \vec{A}_{sc}^{(1)}}{D_0} = \frac{k^2}{4\pi}(\epsilon_r - 1)(\vec{\epsilon}^* \cdot \vec{\epsilon}_0) \times$$

$$\times \underbrace{\int_{r<a} d^3x \, e^{i\vec{q}\vec{x}}}_{I}.$$

$$I = 2\pi \int_0^a dr\, r^2 \int_{-1}^{1} dx \, e^{iqrx} =$$

$$= \frac{4\pi}{q}\int_0^a dr\, r\sin(qr) = \frac{4\pi}{q^3}\left[\sin(qa) - qa\cos(qa)\right].$$

Consequently,

$$\frac{\vec{\epsilon}^{*}\cdot\vec{A}_{sc}^{(1)}}{D_0} = \frac{(ka)^2}{q}(\epsilon_r-1)(\vec{\epsilon}^{*}\cdot\vec{\epsilon}_0)\times$$

$$\times\underbrace{\frac{\sin(qa)-qa\cos(qa)}{(qa)^2}}_{j_1(qa)\text{ - Bessel f'n with } \ell=1}$$

Then $$\frac{d\sigma}{d\Omega} = \left|\frac{\vec{\epsilon}^{*}\cdot\vec{A}_{sc}^{(1)}}{D_0}\right|^2 = k^4a^6|\epsilon_r-1|^2\times$$

$$\times|\vec{\epsilon}^{*}\cdot\vec{\epsilon}_0|^2\left(\frac{j_1(qa)}{qa}\right)^2.$$

The unpolarized cross-section is

$$\frac{d\sigma^{unp}}{d\Omega} = k^4a^6|\epsilon_r-1|^2\left(\frac{j_1(qa)}{qa}\right)^2\frac{1+\cos^2\theta}{2}$$

Note that if $ka \ll 1 \Rightarrow qa \ll 1$,

and $j_1(qa) \to \frac{qa}{3}$ as $qa \to 0$.

So $$\frac{d\sigma^{unp}}{d\Omega} \xrightarrow[qa\ll 1]{} \frac{k^4a^6}{9}|\epsilon_r-1|^2\,\frac{1+\cos^2\theta}{2}.$$

If $ka \gg 1 \Rightarrow qa = 2ka\sin\frac{\theta}{2} \gg 1$, ← true for all angles away from $\theta=0$

and $j_1(qa) \xrightarrow[qa\gg1]{} -\frac{\cos(qa)}{qa}$

So, for θ away from 0

$$\frac{d\sigma^{unp}}{d\Omega} \underset{\nearrow}{\sim} \frac{1}{(qa)^4} = \frac{1}{(2ka\sin\frac{\theta}{2})^4}$$

$qa \gg 1$ falls off rapidly

In σ^{unp}, $\int d\Omega$ will be highly peaked around $\theta = 0$:

$$\sigma^{unp} \sim \int d\Omega \underbrace{\frac{1+\cos^2\theta}{2}}_{\simeq 1} \left(\frac{j_1(qa)}{qa}\right)^2 \simeq$$

$$\simeq 2\pi \int_0^\pi \sin\theta \, d\theta \left(\frac{j_1(qa)}{qa}\right)^2 \simeq 2\pi \int_0^\pi d\theta \, \theta \left(\frac{j_1(ka\theta)}{ka\theta}\right)^2 \simeq$$

$qa \simeq ka\theta$ if θ is small

extend to $+\infty$

$$\simeq \frac{2\pi}{(ka)^2} \int_0^\infty j_1^2(x) \frac{dx}{x} = \frac{\pi}{2(ka)^2}.$$

Finally,

$$\sigma^{unp} \simeq \frac{\pi a^2}{2}(ka)^2 |\epsilon_r - 1|^2, \text{ as desired}$$

Note that $ka \ll 1 \Rightarrow \sigma^{unp} \sim k^4$

$ka \gg 1 \Rightarrow \sigma^{unp} \sim k^2.$

(11.1) a) $y'=y$ (motion in x direction)
$z'=z$

Homogeneity of space-time $\Rightarrow$ linear transform: $\begin{cases} x' = f_1 x + f_2 t \\ t' = g_1 x + g_2 t \end{cases}$

f_1, f_2, g_1, g_2 do not depend on x & t but are otherwise arbitrary functions of v. Inspired by Galilean relativity, we take v out as follows:

$$\begin{cases} x' = f_1 x \overbrace{- v\tilde{f}_2}^{f_2} t \\ t' = g_2 t \underbrace{- v\tilde{g}_1}_{g_1} x \end{cases} \Leftarrow \text{expect } x' = x - vt \text{ as } \frac{v}{c} \to 0$$

But due to homogeneity the points along x cannot spread out as time increases $\Rightarrow f_1 = \tilde{f}_2$.

Moreover, due to isotropy expect $f_1, g_2, \tilde{g}_1$ to be functions of v^2 (no dependence on sign of v)

Finally, $\begin{cases} x' = f_1(v^2) x - f_1(v^2) v t, & y' = y, \\ t' = g_2(v^2) t - \tilde{g}_1(v^2) v x, & z' = z \end{cases}$

The inverse can be obtained by $v \to -v$:

$$\begin{cases} x = f_1(v^2)\,x' + f_1(v^2)\,v\,t', & y = y', \\ t = g_2(v^2)\,t' + \tilde{g}_1(v^2)\,v\,x', & z = z'. \end{cases}$$

b) Transform $K \to K'$ & then $K' \to K$, should recover original coordinates:

$$x = f_1 x' + f_1 v t' = f_1(f_1 x - f_1 v t) +$$

$$+ f_1 v(g_2 t - \tilde{g}_1 v x) =$$

$$= \underbrace{(f_1^2 - f_1 \tilde{g}_1 v^2)}_{=1} x + \underbrace{(f_1 v g_2 - f_1^2 v)}_{=0} t$$

$$\Downarrow \qquad\qquad\qquad \Downarrow$$

$$f_1(f_1 - v^2 \tilde{g}_1) = 1 \qquad\qquad f_1 = g_2$$

Likewise,

$$t = g_2 t' + \tilde{g}_1 v x' = g_2(g_2 t - \tilde{g}_1 v x) +$$

$$+ \tilde{g}_1 v (f_1 x - f_1 v t) =$$

$$= \underbrace{(\tilde{g}_1 v f_1 - \tilde{g}_1 v g_2)}_{=0} x + \underbrace{(g_2^2 - \tilde{g}_1 f_1 v^2)}_{=1} t$$

$$\Downarrow \qquad\qquad\qquad \Downarrow$$

$$f_1 = g_2 \qquad\qquad g_2^2 = 1 + \tilde{g}_1 f_1 v^2$$

$$f_1^2 - \tilde{g}_1 f_1 v^2 = 1,$$ same as above

So now $\begin{cases} \tilde{g}_1 = \dfrac{f_1^2 - 1}{f_1 v^2}, \\ g_2 = f_1 \end{cases}$.

The transform only depends on $f_1(v^2)$.

c) $u = \dfrac{dx}{dt} = \dfrac{f(dx' + v\,dt')}{f\,dt' + v\,dx' \dfrac{f^2-1}{f v^2}}$ $=$

rename $f_1 \to f$ for simplicity

$$u' = \frac{dx'}{dt'}$$

$$= \frac{u' + v}{1 + v u' \underbrace{\dfrac{f - 1/f}{f v^2}}_{\tilde{g}_1/f}}.$$

Universal limiting speed c:

$$c = \frac{c + v}{1 + v c \dfrac{f - 1/f}{f v^2}} \Rightarrow f = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma$$

Finally, $\begin{cases} x' = \gamma(x - vt), & y' = y, \\ t' = \gamma\underbrace{\left(t - \dfrac{vx}{c^2}\right)}_{t - \frac{\beta x}{c}}, & z' = z. \end{cases}$

$$\frac{f^2 - 1}{f v} = \frac{\gamma^2 - 1}{\gamma v} = \frac{\gamma v}{c^2}\left(\frac{\gamma - 1/\gamma}{\gamma \left(\frac{v}{c}\right)^2}\right) = \frac{\gamma v}{c^2} \frac{\gamma^2 - 1}{\gamma^2 \left(\frac{v}{c}\right)^2} = \frac{\gamma v}{c^2}.$$

Likewise,

$$\begin{cases} x = \gamma(x' + vt'), & y = y', \\ t = \gamma\left(t' + \dfrac{vx'}{C^2}\right), & z = z'. \end{cases}$$

$v \to -v$

$C = c$ (speed of light)
by experiments

11.3 Consider two Lorentz boosts along x-axis:

$$A_1 = \begin{pmatrix} \gamma_1 & -\beta_1\gamma_1 & 0 & 0 \\ -\beta_1\gamma_1 & \gamma_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \beta_1 = \frac{v_1}{c} \qquad \gamma_1 = \frac{1}{\sqrt{1-\beta_1^2}}$$

and same for A_2 (with β_2, γ_2).

Then

$$A_2 A_1 = \begin{pmatrix} \gamma_1\gamma_2(1+\beta_1\beta_2) & -\gamma_1\gamma_2(\beta_1+\beta_2) & 0 & 0 \\ -\gamma_1\gamma_2(\beta_1+\beta_2) & \gamma_1\gamma_2(1+\beta_1\beta_2) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Note that

$$\gamma_1\gamma_2(1+\beta_1\beta_2) = \frac{1}{\sqrt{\frac{(1-\beta_1^2)(1-\beta_2^2)}{(1+\beta_1\beta_2)^2}}} =$$

$$= \frac{1}{\sqrt{1-\left(\frac{\beta_1+\beta_2}{1+\beta_1\beta_2}\right)^2}} = \frac{1}{\sqrt{1-\frac{1}{c^2}\Bigg(\underbrace{\frac{v_1+v_2}{1+\frac{v_1 v_2}{c^2}}}_{''v}\Bigg)^2}} =$$

$$= \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \equiv \gamma . \qquad \beta = \frac{v}{c}$$

Likewise,

$$-\gamma_1\gamma_2(\beta_1+\beta_2) = -\frac{\beta_1+\beta_2}{1+\beta_1\beta_2}\overbrace{\gamma_1\gamma_2(1+\beta_1\beta_2)}^{\gamma} =$$

$$= -\frac{v}{c}\gamma = -\beta\gamma$$

Finally,

$$A_2 A_1 = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

This is a Lorentz boost with

$$v = \frac{v_1+v_2}{1+\frac{v_1 v_2}{c^2}}.$$

Note that $v \neq v_1+v_2$ at relativistic speeds