HW #5 solutions

(10.1) a) Recall that for a perfectly conducting sphere,

$$\frac{d\sigma}{d\Omega}(\vec{n},\vec{\epsilon};\vec{n}_0,\vec{\epsilon}_0) = k^4 a^6 \left| \vec{\epsilon}^{\,*}\cdot\vec{\epsilon}_0 - \frac{1}{2}(\vec{n}\times\vec{\epsilon}^{\,*})\cdot(\vec{n}_0\times\vec{\epsilon}_0)\right|^2$$

(10.14)

Here, $\vec{\epsilon}$ is the outgoing polarization. To sum over outgoing polarizations, I'll introduce a basis $\perp \vec{n}$:

$$\begin{cases} \vec{\epsilon}_1 = \dfrac{\vec{n}\times\vec{n}_0}{\sin\theta}, \\ \vec{\epsilon}_2 = \underbrace{\vec{n}\times\vec{\epsilon}_1}_{\perp \vec{\epsilon}_1} = \dfrac{\vec{n}(\vec{n}\cdot\vec{n}_0)-\vec{n}_0}{\sin\theta} \end{cases}$$

$\sin\theta$ is the normalization factor; $\theta = \widehat{\vec{n}\vec{n}_0}$.

Note that $\sin^2\theta = \underbrace{1-(\vec{n}\cdot\vec{n}_0)^2}_{(\vec{n}\times\vec{n}_0)\cdot(\vec{n}\times\vec{n}_0)}$

sum over polar'ns

Now, $$\frac{d\sigma}{d\Omega}(\vec{n};\vec{n}_0,\vec{\epsilon}_0) = \frac{k^4a^6}{1-(\vec{n}\cdot\vec{n}_0)^2}\times$$

$$\times\left\{\left|\underbrace{(\vec{n}\times\vec{n}_0)}_{\vec{\epsilon}_1}\cdot\vec{\epsilon}_0 - \frac{1}{2}(\vec{n}\times(\vec{n}\times\vec{n}_0))\cdot(\vec{n}_0\times\vec{\epsilon}_0)\right|^2 + \left|\underbrace{(\vec{n}(\vec{n}\cdot\vec{n}_0)-\vec{n}_0)}_{\vec{\epsilon}_2}\cdot\vec{\epsilon}_0 - \frac{1}{2}(\vec{n}\times(\vec{n}(\vec{n}\cdot\vec{n}_0)-\vec{n}_0))\cdot(\vec{n}_0\times\vec{\epsilon}_0)\right|^2\right\} =$$

$$\overset{\text{vector algebra using } \vec{n}_0\cdot\vec{\epsilon}_0=0}{=} \frac{k^4 a^6}{1-(\vec{n}\cdot\vec{n}_0)^2}\left[|\vec{n}\cdot(\vec{n}_0\times\vec{\epsilon}_0)|^2 \left[1-\frac{\vec{n}\cdot\vec{n}_0}{2}\right]^2 + |\vec{n}\cdot\vec{\epsilon}_0|^2\left[\frac{1}{2}-(\vec{n}\cdot\vec{n}_0)\right]^2\right].$$

Expand the squares & simplify:

$$\frac{d\sigma}{d\Omega}(\vec{n};\vec{n}_0,\vec{\epsilon}_0) = \frac{k^4 a^6}{1-(\vec{n}\cdot\vec{n}_0)^2} \times$$

$$\times\left[\left(\frac{5}{4}-\vec{n}\cdot\vec{n}_0\right)\left(|\vec{n}\cdot(\vec{n}_0\times\vec{\epsilon}_0)|^2 + |\vec{n}\cdot\vec{\epsilon}_0|^2\right) - \left[1-(\vec{n}\cdot\vec{n}_0)^2\right]\left[\frac{1}{4}|\vec{n}\cdot(\vec{n}_0\times\vec{\epsilon}_0)|^2 + |\vec{n}\cdot\vec{\epsilon}_0|^2\right]\right].$$

To proceed, note that

$\vec{n}_0, \vec{\epsilon}_0, \vec{n}_0\times\vec{\epsilon}_0$ form a basis.

$\vec{n}$ can be expanded in this basis:

$\vec{n}\cdot\vec{n}_0$, $\vec{n}\cdot\vec{\epsilon}_0$, $\vec{n}\cdot(\vec{n}_0\times\vec{\epsilon}_0)$.

$$\vec{n}^2 = 1 \Rightarrow |\vec{n}\cdot(\vec{n}_0\times\vec{\epsilon}_0)|^2 + |\vec{n}\cdot\vec{\epsilon}_0|^2 = 1-(\vec{n}\cdot\vec{n}_0)^2.$$

Finally,

$$\frac{d\sigma}{d\Omega}(\vec{n};\vec{n}_0,\vec{\epsilon}_0) = k^4 a^6\left[\frac{5}{4}-\vec{n}\cdot\vec{n}_0 - |\vec{n}\cdot\vec{\epsilon}_0|^2 - \frac{1}{4}|\vec{n}\cdot(\vec{n}_0\times\vec{\epsilon}_0)|^2\right],$$ as desired

b) As in (a), expand $\vec{n}$ in the $\vec{n}_0, \vec{\epsilon}_0, \vec{n}_0 \times \vec{\epsilon}_0$ basis:

$$\begin{cases} \vec{n}\cdot\vec{n}_0 = \cos\theta, \\ \vec{n}\cdot\vec{\epsilon}_0 = \sin\theta\cos\varphi, \\ \vec{n}\cdot(\vec{n}_0\times\vec{\epsilon}_0) = \sin\theta\sin\varphi. \end{cases}$$

Then

$$\frac{d\sigma}{d\Omega}(\theta,\varphi) \underset{\text{from (a)}}{=} k^4 a^6 \left[\frac{5}{4} - \cos\theta - \sin^2\theta\cos^2\varphi - \frac{1}{4}\sin^2\theta\sin^2\varphi\right] =$$

$$= k^4 a^6 \left[\frac{5}{8}(1+\cos^2\theta) - \cos\theta - \frac{3}{8}\sin^2\theta\cos 2\varphi\right].$$

c) First, $\frac{d\sigma}{d\Omega}(\frac{\pi}{2},\varphi) = k^4 a^6\left[\frac{5}{8} - \frac{3}{8}\cos 2\varphi\right]$.

Then $\begin{cases} \frac{d\sigma}{d\Omega}(\frac{\pi}{2},0) = \frac{k^4 a^6}{4}, \\ \frac{d\sigma}{d\Omega}(\frac{\pi}{2},\frac{\pi}{2}) = k^4 a^6. \end{cases}$

The ratio is $1/4$.

$$\theta=\frac{\pi}{2}, \varphi=0 \Rightarrow \begin{cases} \vec{n}\cdot\vec{n}_0 = 0, \\ \vec{n}\cdot\vec{\epsilon}_0 = 1, \\ \vec{n}\cdot(\vec{n}_0\times\vec{\epsilon}_0) = 0 \end{cases} \Rightarrow \vec{n}\uparrow\uparrow\vec{\epsilon}_0$$

This means that the induced electric dipole (which is $\uparrow\uparrow \vec{E}_0$), is $\uparrow\uparrow \vec{n}$ as well. But the dipole does not radiate along its axis, so radiation is purely magnetic dipole in this case.

Similarly, at $\theta = \frac{\pi}{2}$, $\varphi = \frac{\pi}{2}$ radiation is purely electric dipole in nature. The factor of $1/4$ indicates the relative strength of magnetic dipole / electric dipole radiation.

(10.4) a) $R \ll \lambda \Rightarrow$ we are again in the quasistatic limit of a sphere in an external uniform field.

The induced electric dipole moment is:

$$\vec{p} = 4\pi\epsilon_0 \left(\frac{\epsilon_r^c - 1}{\epsilon_r^c + 2} \right) R^3 \vec{\epsilon}_0 E_0$$

(10.5)

Since the sphere is only slightly lossy, we can neglect the induced magnetic dipole moment.

Then $\underset{\text{complex}}{\epsilon_r^c} = \underset{\text{real}}{\epsilon_r} + i \frac{\overset{\text{real}}{\sigma}}{\omega \epsilon_0}$

$$\omega = ck = \frac{k}{\epsilon_0 Z_0} \Rightarrow \epsilon_r^c = \epsilon_r + i \frac{\sigma}{k} Z_0$$

Now recall that

$$\frac{d\sigma}{d\Omega} = \frac{k^4}{(4\pi\epsilon_0)^2 E_0^2} |\vec{\epsilon}^* \cdot \vec{p}|^2 \text{, where}$$

(10.4)

$$\vec{p} = 4\pi\epsilon_0 \frac{(\epsilon_r - 1) + i \frac{\sigma Z_0}{k}}{(\epsilon_r + 2) + i \frac{\sigma Z_0}{k}} R^3 \vec{\epsilon}_0 E_0 .$$

Then $$\frac{d\sigma}{d\Omega} = \frac{(\epsilon_r - 1)^2 + \left(\frac{Z_0 \sigma}{k}\right)^2}{(\epsilon_r + 2)^2 + \left(\frac{Z_0\sigma}{k}\right)^2} k^4 R^6 |\vec{\epsilon}^* \cdot \vec{\epsilon}_0|^2 .$$

For unpolarized light, we have

$$\frac{d\sigma_{unpol}}{d\Omega} = \frac{(\epsilon_r-1)^2 + \left(\frac{Z_0\sigma}{k}\right)^2}{(\epsilon_r+2)^2 + \left(\frac{Z_0\sigma}{k}\right)^2} k^4 R^6 \frac{1+\cos^2\theta}{2}$$

(10.10)

Finally,

$$\sigma_{unpol} = \int d\Omega \frac{d\sigma}{d\Omega} = \frac{8\pi}{3} \frac{(\epsilon_r-1)^2 + \left(\frac{Z_0\sigma}{k}\right)^2}{(\epsilon_r+2)^2 + \left(\frac{Z_0\sigma}{k}\right)^2} k^4 R^6 .$$

(10.11)