7.19 $\qquad u(x,0) = \underset{\substack{\uparrow \\ \text{modulation} \\ \text{envelope}}}{f(x)}\, e^{ik_0 x}$

(a) $f(x) = N e^{-\alpha|x|/2}$.

Recall that $A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,0)\, e^{-ikx}\, dx$.

Then
$$A(k) = \frac{N}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, e^{i(k_0-k)x} e^{-\alpha|x|/2} =$$

$$= \frac{2N}{\sqrt{2\pi}} \int_0^{\infty} dx\, \underset{\substack{\uparrow \\ \text{solveable}}}{\cos[(k-k_0)x]}\, e^{-\alpha x/2} = \frac{N\alpha}{\sqrt{2\pi}\left(\frac{\alpha^2}{4} + (k-k_0)^2\right)}.$$

$$\int_{-\infty}^{\infty} \Rightarrow \int_0^{\infty} + \underbrace{\int_{-\infty}^{0}}_{x \to -x}$$

So,
$$\begin{cases} |A(k)|^2 = \dfrac{N^2\alpha^2}{2\pi\left(\frac{\alpha^2}{4} + (k-k_0)^2\right)^2}, \\ |u(x,0)|^2 = N^2 e^{-\alpha|x|}. \end{cases}$$

Finally, recall that
$$\Delta x^2 = \frac{\int_{-\infty}^{\infty} dx\, x^2 |f(x)|^2}{\int_{-\infty}^{\infty} dx\, \underbrace{|f(x)|^2}_{|u(x,0)|^2}},$$
& similarly for Δk^2.

Then
$$\Delta x^2 = \frac{\int_{-\infty}^{\infty} dx\, x^2 e^{-\alpha|x|}}{\int_{-\infty}^{\infty} dx\, e^{-\alpha|x|}} \Rightarrow \Delta x = \frac{\sqrt{2}}{\alpha}$$
(units OK)

$$\Delta k^2 = \frac{\int_{-\infty}^{\infty} dk\,(k-k_0)^2 \frac{1}{\left[\frac{\alpha^2}{4}+(k-k_0)^2\right]^2}}{\int_{-\infty}^{\infty} dk\, \frac{1}{\left[\frac{\alpha^2}{4}+(k-k_0)^2\right]^2}} \Rightarrow \Delta k = \frac{\alpha}{2}$$

So, $\Delta x \Delta k = \frac{1}{\sqrt{2}} \geq \frac{1}{2}$.

(b) $f(x) = N e^{-\frac{\alpha^2 x^2}{4}}$

Similarly,

$$A(k) = \frac{2N}{\sqrt{2\pi}} \int_0^{\infty} dx \cos[(k-k_0)x]\, e^{-\frac{\alpha^2 x^2}{4}} =$$

$$= \sqrt{\frac{2}{\alpha^2}}\, N e^{-\frac{(k-k_0)^2}{\alpha^2}}$$

Then $\begin{cases} |A(k)|^2 = \frac{2N^2}{\alpha^2} e^{-\frac{2(k-k_0)^2}{\alpha^2}}, \\ |u(x,0)|^2 = N^2 e^{-\frac{\alpha^2 x^2}{2}}. \end{cases}$

$$\Delta x^2 = \frac{\int_{-\infty}^{\infty} dx\, x^2 e^{-\frac{\alpha^2 x^2}{2}}}{\int_{-\infty}^{\infty} dx\, e^{-\frac{\alpha^2 x^2}{2}}} \Rightarrow \Delta x = \frac{1}{\alpha}$$

$$\Delta k^2 = \frac{\int_{-\infty}^{\infty} dk\,(k-k_0)^2 e^{-\frac{2(k-k_0)^2}{\alpha^2}}}{\int_{-\infty}^{\infty} dk\, e^{-\frac{2(k-k_0)^2}{\alpha^2}}} \Rightarrow \Delta k = \frac{\alpha}{2}$$

$\Delta x \Delta k = \frac{1}{2}$, minimum possible.

(c) $$f(x) = \begin{cases} N(1-\alpha|x|) & \alpha|x| \le 1 \\ 0 & \alpha|x| > 1 \end{cases}$$

$$\Delta x^2 = \frac{\int_{-1/\alpha}^{1/\alpha} dx\, x^2 (1-\alpha|x|)^2}{\int_{-1/\alpha}^{1/\alpha} dx\, (1-\alpha|x|)^2} = \frac{1/(15\alpha^3)}{2/(3\alpha)} =$$

$$= \frac{1}{10\alpha^2}. \quad \Rightarrow \quad \Delta x = \frac{1}{\sqrt{10}\,\alpha}.$$

Further, $$A(k) = \frac{2N}{\sqrt{2\pi}} \int_0^{1/\alpha} dx \cos[(k-k_0)x](1-\alpha x) =$$

$$\underset{\text{by parts}}{=} \frac{2N}{\sqrt{2\pi}} \frac{\alpha}{(k-k_0)^2} \left[1 - \cos\left(\frac{k-k_0}{\alpha}\right)\right].$$

Then $$\Delta k^2 = \frac{\int_{-\infty}^{\infty} dk \frac{[1-\cos(\frac{k}{\alpha})]^2}{k^2}}{\int_{-\infty}^{\infty} dk \frac{[1-\cos(\frac{k}{\alpha})]^2}{k^4}} = \alpha^2 \frac{\pi}{\pi/3} = 3\alpha^2.$$

So, $\Delta k = \sqrt{3}\,\alpha$

$$\Delta x \, \Delta k = \sqrt{\frac{3}{10}} \simeq 0.55 > \frac{1}{2}.$$

(d) $f(x) = \begin{cases} N & |x| < a \\ 0 & |x| > a \end{cases}$

$$A(k) = \frac{N}{\sqrt{2\pi}} \int_{-a}^{a} dx\, e^{-i(k-k_0)x} = \frac{2N}{\sqrt{2\pi}} \frac{\sin[(k-k_0)a]}{k-k_0}.$$

$|A(k)|^2$ & $|u(x,0)|^2$ clear from above.

$$\Delta x^2 = \frac{\int_{-a}^{a} dx\, x^2}{\int_{-a}^{a} dx} \Rightarrow \Delta x = \frac{a}{\sqrt{3}}$$

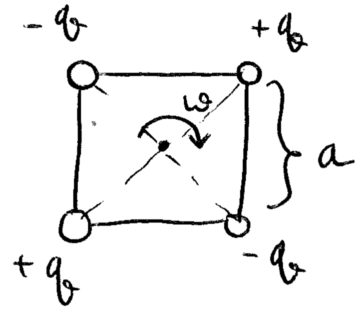
divergés

$$\Delta k^2 = \frac{\int_0^\infty dk \sin^2(ka)}{\int_0^\infty dk \frac{\sin^2(ka)}{k^2}} \Rightarrow \Delta k = \infty$$

finite

$$\Delta x \Delta k = \infty \geq \frac{1}{2}$$

9.2



Choose the z-axis as the axis of rotation. Choose $t=0$ s.t. the phase is ωt for the $+q$ charge in the upper right corner.

Then
$$\rho(\vec{x},t) = q\,\delta(z)\Big\{\delta(x-\underbrace{\tfrac{a}{\sqrt{2}}}_{\frac{1}{2}\text{ the diagonal}}\cos\omega t)\,\delta(y-\tfrac{a}{\sqrt{2}}\sin\omega t)-$$
$$-\delta(x-\tfrac{a}{\sqrt{2}}\cos(\omega t+\tfrac{\pi}{2}))\,\delta(y-\tfrac{a}{\sqrt{2}}\sin(\omega t+\tfrac{\pi}{2}))+$$
$$+\delta(x-\tfrac{a}{\sqrt{2}}\cos(\omega t+\pi))\,\delta(y-\tfrac{a}{\sqrt{2}}\sin(\omega t+\pi))-$$
$$-\delta(x-\tfrac{a}{\sqrt{2}}\cos(\omega t+\tfrac{3\pi}{2}))\,\delta(y-\tfrac{a}{\sqrt{2}}\sin(\omega t+\tfrac{3\pi}{2}))\Big\}=$$

$$= q\,\delta(z)\Big\{\delta(x-\tfrac{a}{\sqrt{2}}\cos\omega t)\,\delta(y-\tfrac{a}{\sqrt{2}}\sin\omega t)-$$
$$-\delta(x+\tfrac{a}{\sqrt{2}}\sin\omega t)\,\delta(y-\tfrac{a}{\sqrt{2}}\cos\omega t)+$$
$$+\delta(x+\tfrac{a}{\sqrt{2}}\cos\omega t)\,\delta(y+\tfrac{a}{\sqrt{2}}\sin\omega t)-$$
$$-\delta(x-\tfrac{a}{\sqrt{2}}\sin\omega t)\,\delta(y+\tfrac{a}{\sqrt{2}}\cos\omega t)\Big\}$$

Recall that $Q_{\alpha\beta} = \int dV\,\rho(\vec{x},t)\,[3x_\alpha x_\beta - r^2\delta_{\alpha\beta}]$

Now, $\int dV\,\rho(\vec{x},t)\,z^2 = 0$,

$$\int dV\,\rho(\vec{x},t)\,x^2 = q\Big\{\frac{a^2}{2}\cos^2\omega t - \frac{a^2}{2}\sin^2\omega t + $$
$$+\frac{a^2}{2}\cos^2\omega t - \frac{a^2}{2}\sin^2\omega t\Big\} =$$

$$= q a^2\cos(2\omega t).$$

Likewise, $\int dV\, \rho(\vec{x},t)\, y^2 = -q a^2 \cos(2\omega t)$.

$$\int dV\, \rho(\vec{x},t)\, xz = \int dV\, \rho(\vec{x},t)\, yz = 0 .$$

Finally, $\int dV\, \rho(\vec{x},t)\, xy = q \Big\{ \frac{a^2}{2} \sin\omega t \cos\omega t +$

$$+ \frac{a^2}{2} \sin\omega t \cos\omega t + \frac{a^2}{2} \sin\omega t \cos\omega t +$$

$$+ \frac{a^2}{2} \sin\omega t \cos\omega t \Big\} = q a^2 \sin(2\omega t).$$

Thus $Q_{11} = \int dV\, \rho\, (2x^2 - y^2 - z^2) = 3 q a^2 \cos(2\omega t) =$

$$= 3 q a^2 \, \mathrm{Re}\{ e^{-2i\omega t} \}$$

$$Q_{22} = \int dV\, \rho\, (2y^2 - x^2 - z^2) = -3 q a^2 \cos(2\omega t) =$$

$$= -3 q a^2 \, \mathrm{Re}\{ e^{-2i\omega t} \}$$

$$Q_{12} = Q_{21} = \int dV\, \rho\, xy = 3 q a^2 \sin(2\omega t) =$$

$$= 3 q a^2 \, \mathrm{Re}\{ i e^{-2i\omega t} \}.$$

All other $Q_{\alpha\beta} = 0$.

Therefore, $Q_{\alpha\beta} = 3 q a^2 \begin{pmatrix} 1 & i & 0 \\ i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} e^{-2i\omega t}$

The freq. of radiation is 2ω $\left(k = \frac{2\omega}{c}\right)$, consistent with the symmetry of the charge distribution.

This system has no electric dipole moment (2 ↑↓ dipoles), no magnetic dipole moment ($\langle\vec{J}\rangle = 0$). Thus radiation is
$\uparrow$ average over $T = \frac{2\pi}{\omega}$

dominated by the electric quadrupole:

$$\vec{H} = -\frac{ick^3}{24\pi}\frac{e^{ikr}}{r}\,\vec{n}\times\vec{Q}(\vec{n})$$

(9.44)

$$\vec{n} = (\sin\theta\cos\varphi,\ \sin\theta\cos\varphi,\ \cos\theta)$$

$$\vec{Q}(\vec{n}) = (Q_{1\alpha}n_\alpha,\ Q_{2\alpha}n_\alpha,\ Q_{3\alpha}n_\alpha) =$$

$$= \Big(3qa^2\sin\theta(\cos\varphi + i\sin\varphi),\ 3qa^2\sin\theta(i\cos\varphi - \sin\varphi),\ 0\Big) = \underline{\underline{3qa^2\sin\theta\, e^{i\varphi}(\hat{x} + i\hat{y})}}.$$

Thus
$$\begin{cases} \vec{H} = -\dfrac{ick^3}{8\pi}(qa^2)\dfrac{e^{ikr}}{r}e^{i\varphi}\sin\theta\ * \\ \quad *\,(-i\cos\theta,\ \cos\theta,\ i\sin\theta\, e^{i\varphi}), \\ \vec{E} = Z_0\vec{H}\times\vec{n}. \end{cases}$$

[$e^{-2i\omega t}$ dependence is implied]

Finally,
$$\frac{dP}{d\Omega} = \frac{c^2 Z_0}{1152\pi^2}k^6\,\underbrace{|(\vec{n}\times\vec{Q})\times\vec{n}|^2}_{|\vec{Q}|^2 - |\vec{Q}\cdot\vec{n}|^2} =$$

(9.45)

$$\stackrel{(=)}{} \frac{c^2 Z_0}{1152\pi^2} k^6 (3qa^2)^2 \underbrace{\sin^2\theta(2-\sin^2\theta)}_{\begin{subarray}{l}(1-\cos^2\theta)(1+\cos^2\theta) = \\ = (1-\cos^4\theta)\end{subarray}} =$$

$$= \frac{c^2 Z_0 k^6 (qa^2)^2}{128\pi^2} (1-\cos^4\theta) .$$

$$k = \frac{2\omega}{c}$$

The total radiated power is

$$P = \int d\Omega \frac{dP}{d\Omega} = \frac{Z_0 \omega^6}{2\pi^2 c^4} (qa^2)^2 \int d\Omega (1-\cos^4\theta) =$$

$$= \frac{8 Z_0 \omega^6}{5\pi c^4} q^2 a^4 .$$

(9.3) This setup (two half-spheres separated by a gap) will produce a non-zero electric dipole, which will dominate in the $kd \gg 1$ limit.

Eq'n just above (3.38):

$$\varphi(r,\theta,t) = \frac{V(t)}{\sqrt{\pi}} \sum_{j=1}^{\infty} (-1)^{j+1} \frac{(2j-\frac{1}{2})\Gamma(j-\frac{1}{2})}{j!} \left(\frac{a}{r}\right)^{2j} P_{2j-1}(\cos\theta),$$

↑ scalar potential

where a - sphere radius, and $P_\ell(x)$ is the Legendre polynomial of order ℓ.

$j=1$ is the electric dipole term:

$$\varphi_1(r,\theta,t) = \frac{V(t)}{\sqrt{\pi}} \frac{3}{2} \underbrace{\Gamma\left(\tfrac{1}{2}\right)}_{=\sqrt{\pi}} \left(\frac{R}{r}\right)^2 \underbrace{P_1(\cos\theta)}_{=\cos\theta} =$$

sphere radius

$$= \frac{3V}{2}\left(\frac{R}{r}\right)^2 \cos\theta \cos\omega t \equiv$$

$$\equiv \frac{p}{4\pi\epsilon_0 r^2} \cos\theta \cos\omega t,$$

← scalar potential due to dipole pointing in $\hat{z}$-direction

s.t. the dipole moment

$$p = 6\pi\epsilon_0 V R^2, \quad \vec{p} = p\hat{z}.$$

$\hat{z}$ is $\perp$ to the plane of the insulating gap.

For electric dipole radiation,
everything is known (see Ch. 9):

radiation zone

$k = \frac{\omega}{c}$

$$\begin{cases} \vec{H} = -\frac{3V}{2Z_0}(kR)^2 \frac{e^{ikr}}{r} \sin\theta\, \hat{\varphi}, \\ \vec{E} = -\frac{3V}{2}(kR)^2 \frac{e^{ikr}}{r} \sin\theta\, \hat{\theta}, \\ \frac{dP}{d\Omega} = \frac{9V^2}{8Z_0}(kR)^4 \sin^2\theta, \\ P = 3\pi (kR)^4 \frac{V^2}{Z_0}. \end{cases}$$