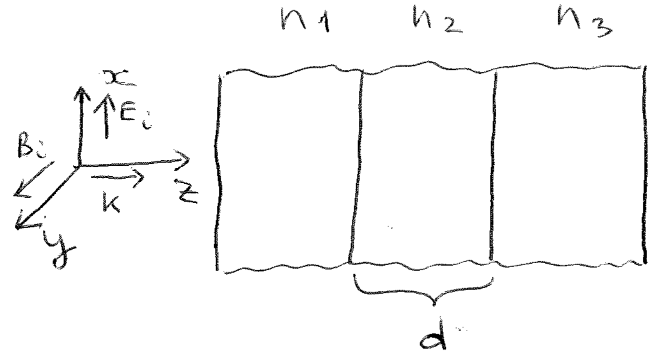
HW #3 solutions

7.2



Non-magnetic media

(a) Choose a coordinate system s.t.

$$\begin{cases} \vec{E}_i = E_i e^{i(k_1 z - \omega t)} \hat{x}, \\ \vec{B}_i = \underbrace{\frac{k_1 E_i}{\omega}}_{\frac{E_i}{v_1}} e^{i(k_1 z - \omega t)} \hat{y} \end{cases}$$

incident wave

, v_1 - phase velocity in medium n_1

k_1 - wavenumber in medium n_1

Likewise,

$$\begin{cases} \vec{E}_r = E_r e^{i(-k_1 z - \omega t)} \hat{x}, \\ \vec{B}_r = -\frac{E_r}{v_1} e^{-i(k_1 z - \omega t)} \hat{y} \end{cases}$$

reflected wave

Note that multiple reflections from both boundaries are included into E_r.

In medium n_2, there are 2 waves as well:

$$\begin{cases} \vec{E}_+ = E_+ e^{i(k_2 z - \omega t)} \hat{x}, \\ \vec{B}_+ = \frac{E_+}{v_2} e^{i(k_2 z - \omega t)} \hat{y} \end{cases}$$

propagates to the right

$$\begin{cases} \vec{E}_- = E_- e^{i(-k_2 z - \omega t)} \hat{x}, \\ \vec{B}_- = -\frac{E_-}{v_2} e^{-i(k_2 z - \omega t)} \hat{y} \end{cases}$$

propagates to the left

Again, E_+, E_- include multiple reflection events.

Finally, in medium n_3 we have:

$$\begin{cases} \vec{E}_t = E_t\, e^{i(k_3 z - \omega t)}\,\hat{x}, \\ \vec{B}_t = \dfrac{E_t}{v_3}\, e^{i(k_3 z - \omega t)}\,\hat{y} \end{cases}.$$

There is no reflected wave.

Now, use BCs:

$$\begin{cases} \vec{E}^{\parallel}(0^-) = \vec{E}^{\parallel}(0^+), \\ \vec{B}^{\parallel}(0^-) = \vec{B}^{\parallel}(0^+). \end{cases}$$

Here, 0 & d are z-coordinates of the two interfaces

$$\begin{cases} \vec{E}^{\parallel}(d^-) = \vec{E}^{\parallel}(d^+) \\ \vec{B}^{\parallel}(d^-) = \vec{B}^{\parallel}(d^+) \end{cases}$$

$\underline{x=0}$:

$$\begin{cases} E_i + E_r = E_+ + E_-, \\ \dfrac{E_i - E_r}{v_1} = \dfrac{E_+ - E_-}{v_2}. \end{cases}$$

$\underline{x=d}$:

$$\begin{cases} E_+ e^{ik_2 d} + E_- e^{-ik_2 d} = E_t e^{ik_3 d}, \\ \dfrac{E_+ e^{ik_2 d} - E_- e^{-ik_2 d}}{v_2} = \dfrac{E_t}{v_3} e^{ik_3 d} \end{cases}$$

The rest is algebra ...

Define

$$\begin{cases} a_1 = \dfrac{v_1}{v_2} = \dfrac{c/n_1}{c/n_2} = \dfrac{n_2}{n_1}, \\ a_2 = \dfrac{v_2}{v_3} = \dfrac{n_3}{n_2} \end{cases}$$

Then

$$\begin{cases} E_i + E_r = E_+ + E_-, & (1) \\ E_i - E_r = a_1 (E_+ - E_-) & (2) \end{cases}$$

$$\begin{cases} E_+ e^{ik_2 d} + E_- e^{-ik_2 d} = E_t e^{ik_3 d}, & (3) \\ E_+ e^{ik_2 d} - E_- e^{-ik_2 d} = a_2 E_t e^{ik_3 d} & (4) \end{cases}$$

Eqs. (3) & (4) give:

$$\begin{cases} E_+ e^{ik_2 d} = \dfrac{1+a_2}{2} E_t e^{ik_3 d}, \\ E_- e^{-ik_2 d} = \dfrac{1-a_2}{2} E_t e^{ik_3 d} \end{cases}$$

Then Eqs. (1) & (2) give:

$$2E_i = (1+a_1)E_+ + (1-a_1)E_- =$$

$$= \frac{(1+a_1)(1+a_2)}{2} E_t e^{i(k_3-k_2)d} +$$

$$+ \frac{(1-a_1)(1-a_2)}{2} E_t e^{i(k_3+k_2)d}.$$

Then

$$\frac{E_i}{E_t} = \frac{1}{2} e^{ik_3 d} \left\{ \frac{(1+a_1)(1+a_2)}{2} e^{-ik_2 d} + \frac{(1-a_1)(1-a_2)}{2} e^{ik_2 d} \right\} =$$

$$= \frac{e^{ik_3 d}}{2} \left\{ (1+a_1 a_2)\cos(k_2 d) - i(a_1+a_2)\sin(k_2 d) \right\}.$$

Then $4\left|\dfrac{E_i}{E_t}\right|^2 = (1+a_1 a_2)^2 \cos^2(k_2 d) + (a_1+a_2)^2 \sin^2(k_2 d) =$

$$= (1+a_1 a_2)^2 - (1-a_1^2)(1-a_2^2)\sin^2(k_2 d). \quad (*)$$

Finally, the transm'n coeff.

$$T = \frac{I_t}{I_i} = \frac{\epsilon_3 v_3 |E_t|^2}{\epsilon_1 v_1 |E_i|^2} = \frac{\overbrace{n_3}^{a_1 a_2}}{n_1} \frac{|E_t|^2}{|E_i|^2}$$

$$\underset{\uparrow (*)}{=} \frac{4 a_1 a_2}{(1+a_1 a_2)^2 - (1-a_1^2)(1-a_2^2)\sin^2(k_2 d)}$$

In other words,

$$T = \frac{4 n_1 n_2^2 n_3}{n_2^2 (n_1+n_3)^2 + (n_2^2 - n_3^2)(n_2^2 - n_1^2) \sin^2\left(n_2 \frac{\omega}{c} d\right)}.$$

The reflection coeff. $R = 1 - T$.
These (T & R) can be plotted as a f'n of ω.

What happens if $d=0$?

$$\begin{cases} T = \dfrac{4 n_1 n_3}{(n_1+n_3)^2}, \\[2ex] R = 1 - T = \dfrac{n_1^2 + 2n_1 n_3 + n_3^2 - 4 n_1 n_3}{(n_1+n_3)^2} = \dfrac{(n_1-n_3)^2}{(n_1+n_3)^2}. \end{cases}$$

Single interface limit (no n_2 dependence as expected)

(b) $n_3 = 1$; find n_2 & d so there is no reflection @ ω_0.

If $n_3 = 1$,

$$R = \frac{n_2^2 (n_1-1)^2 + (n_2^2-1)(n_2^2 - n_1^2) \sin^2\left(n_2 \frac{\omega}{c} d\right)}{n_2^2 (n_1+1)^2 + (n_2^2-1)(n_2^2-n_1^2)\sin^2\left(n_2 \frac{\omega}{c} d\right)}$$

We require

$$\underbrace{n_2^2 (n_1-1)^2}_{>0\ (n_1>1)} + \underbrace{(n_2^2 - 1)}_{\substack{>0 \\ (n_2>1)}} \underbrace{(n_2^2 - n_1^2)}_{\substack{\text{may be} \\ <0 \text{ if} \\ n_2<n_1}} \underbrace{\sin^2\left(n_2 \frac{\omega_0}{c} d\right)}_{>0} = 0.$$

This is one eq'n for 2 unknowns: n_2 & d $\Rightarrow$

$\Rightarrow$ multiple solutions

One possibility:

$$\begin{cases} \sin^2(\dots) = 1, & \Rightarrow d = \dfrac{c}{n_2 \omega_0} \underbrace{\pi\left(i + \tfrac{1}{2}\right)}_{\sin = \pm 1} \\ n_2^2(n_1-1)^2 + (n_2^2-1)(n_2^2-n_1^2) = 0 & \end{cases}$$

i ← integer

$$\Downarrow \; n_2 \overset{?}{=} \sqrt{n_1}$$

$$n_1(n_1-1)^2 + (n_1-1)(n_1-n_1^2) =$$

$$= n_1(n_1-1)^2 - n_1(n_1-1)^2 = 0$$

So, $\begin{cases} n_2 = \sqrt{n_1}, \\ d = \dfrac{c}{\sqrt{n_1}\,\omega_0}\pi\left(i+\tfrac{1}{2}\right) \end{cases}$ should work

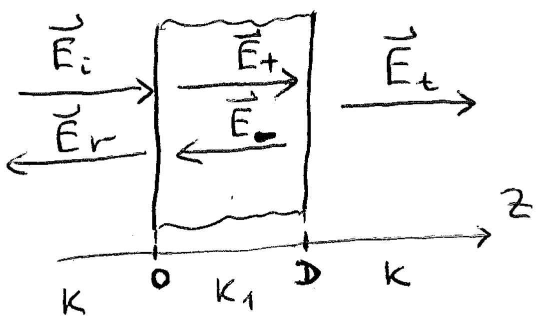
7.5 $\vec{E} = \vec{E}_i \, e^{i(\vec{k}\vec{x} - \omega t)}$; normal incidence

Conductor with $\sigma \gg \omega \epsilon_0$,
thickness D.

a) Recall that

$$\epsilon = \epsilon_0 + i \frac{\sigma}{\omega}$$

Use BCs as in 7.2:



$\underline{z=0}$: $$\begin{cases} E_i + E_r = E_+ + E_i , \\ E_i - E_r = n(E_+ - E_-) . \end{cases} \qquad (*)$$

$$n = \sqrt{\frac{\epsilon}{\epsilon_0}} = \sqrt{1 + i \frac{\sigma}{\omega \epsilon_0}}$$

$\underline{z=D}$: $$\begin{cases} E_+ e^{i\varphi} + E_- e^{-i\varphi} = E_t e^{ikD} , \\ n(E_+ e^{i\varphi} - E_- e^{-i\varphi}) = E_t e^{ikD} . \end{cases} \qquad (**)$$

Note that

$$\varphi = k_1 D = \frac{\omega n}{c} D = \frac{\omega}{c} D \sqrt{1 + i \frac{\sigma}{\omega \epsilon_0}} .$$

If $\frac{\sigma}{\omega\epsilon_0} \gg 1$, $n \simeq \sqrt{i\frac{\sigma}{\omega\epsilon_0}} = \frac{2}{\gamma}$,

where $\gamma = \sqrt{\frac{2\epsilon_0\omega}{\sigma}}(1-i)$.

Moreover,

$$\varphi = \frac{\omega n}{c} D \simeq \frac{\omega}{c} D \frac{2}{\gamma} = i\underbrace{\lambda}_{(1-i)\frac{D}{\delta}}, \quad \delta = \sqrt{\frac{2}{\omega\mu\sigma}} \text{ is the penetr'n depth}$$

Thus $|n| \gg 1 \Rightarrow |\gamma| \ll 1$.

We can use eq's $(*)$ & $(**)$ to express $\frac{E_r}{E_i}$ & $\frac{E_t}{E_i}$ in terms of γ & λ:
(best done in Mathematica)

$$\begin{cases} \dfrac{E_r}{E_i} = \dfrac{e^{-2\lambda}-1}{(1-e^{-2\lambda})+\gamma(1+e^{-2\lambda})}, \\[2ex] \dfrac{E_t}{E_i} = e^{-ikD}\dfrac{2\gamma e^{-\lambda}}{(1-e^{-2\lambda})+\gamma(1+e^{-2\lambda})} \end{cases}$$

Note that all $O(\gamma^2)$ terms have been dropped. Also, there is an overall
(but could've been kept in principle)

phase factor $\sim e^{-ikD}$ which appears to have been dropped in Jackson (it does not affect $\left|\frac{E_t}{E_i}\right|^2$ anyway)

b) $D \to 0$ implies $\lambda \to 0$.

Then $\begin{cases} \dfrac{E_r}{E_i} \to 0, \\ \dfrac{E_t}{E_i} \to 1 \end{cases}$, as expected.

$D \to \infty \Rightarrow \lambda \to \infty$:

$$\begin{cases} \dfrac{E_t}{E_i} \to 0, \\ \dfrac{E_r}{E_i} \to -\dfrac{1}{1+\gamma}. \end{cases}$$

Note that some power is dissipated in the conductor. If $\sigma \to \infty \Rightarrow$ $\Rightarrow \gamma \to 0 \Rightarrow \dfrac{E_r}{E_i} \to -1$, everything gets reflected in the perfect conductor.

c) "Very small thickness": this is about simplifying the denominator in $\dfrac{E_t}{E_i}$. We cannot say that

$$(1-e^{-2\lambda}) + \gamma(1+e^{-2\lambda}) \approx$$

$$\approx 1-e^{-2\lambda} \quad \text{IF}$$

$$|1-e^{-2\lambda}| \approx |\gamma(1+e^{-2\lambda})|.$$

D is small $\Rightarrow \lambda$ is small, s.t.

$|2\lambda| \approx |2\gamma|$, or $\frac{D}{\delta} \approx \frac{\omega\delta}{c}$,

$$D \approx \frac{\omega\delta^2}{c}.$$

So, $D \lesssim \frac{\omega\delta^2}{c}$ is the small thickness limit.

If $D > \frac{\omega\delta^2}{c}$, $\frac{E_t}{E_i} \approx e^{-ikD} \frac{2\gamma e^{-\lambda}}{1-e^{-2\lambda}}$.

Then $T = \left|\frac{E_t}{E_i}\right|^2 = \frac{4\gamma\gamma^* e^{-\lambda+\lambda^*}}{(1-e^{-2\lambda})(1-e^{-2\lambda^*})}$.

Note that $|\gamma|^2 = 2\frac{2\epsilon_0\omega}{\sigma} =$

$= 2(\mathrm{Re}(\gamma))^2$.

Also, $e^{-\lambda} = e^{-D/\delta} e^{iD/\delta}$.

Then

$$T = \frac{8(\mathrm{Re}(\gamma))^2 e^{-2D/\delta}}{1+e^{-4D/\delta} - 2e^{-2D/\delta}\cos(2D/\delta)}.$$

7.4 σ; ϵ free space / medium interface

(a) Normal incidence:

$$\frac{\overset{\text{reflected}}{E_0''}}{\underset{\text{incident}}{E_0}} = \frac{1-n}{1+n}$$

Recall that $n = c/v = c\sqrt{\mu\epsilon}$.

We have argued that for a conductor, we can neglect $\epsilon_b(\omega)$, yielding

$$\epsilon \simeq \frac{i\sigma}{\omega}$$ (true for low ω, or large σ)

Then $n = c\sqrt{i\frac{\mu\sigma}{\omega}} = (1+i)\underbrace{\frac{c}{\omega}}_{k^{-1}}\underbrace{\sqrt{\frac{\mu\sigma\omega}{2}}}_{\delta^{-1}} =$

$$\sqrt{i} = \frac{1+i}{\sqrt{2}}$$

$= (1+i)\frac{1}{k\delta}$, where $[\delta] = L$,

δ - skin depth.

Finally, $$\frac{E_0''}{E_0} = \frac{1-(1+i)\frac{1}{k\delta}}{1+(1+i)\frac{1}{k\delta}} \equiv \underbrace{re^{i\varphi}}_{\text{complex}}$$

After some algebra,

$$\begin{cases} r = \dfrac{\sqrt{\omega^4\delta^4 + 4c^4}}{2c^2 + 2c\omega\delta + \omega^2\delta^2}, \\ \tan\varphi = -\dfrac{2c\omega\delta}{\omega^2\delta^2 - 2c^2}. \end{cases}$$

(b)

Note that $\delta = \sqrt{\frac{2}{\mu\sigma\omega}} \to 0$ as $\sigma \to +\infty$ (good conductor)

Then $r \to 1$, $\tan\varphi \to 0^+ \Rightarrow \varphi \to \pi$

phase change due to $n>1$

The reflected wave must have a phase change since $n > 1$.

For a good conductor,

$$R = r^2 = \frac{\omega^4\delta^4 + 4c^4}{(2c^2 + 2c\omega\delta + \omega^2\delta^2)^2} \approx$$

expand to $O(\delta)$

$$\approx 1 - 2\frac{\omega}{c}\delta .$$

For a poor conductor,

$\sigma \to 0$ & $\delta \to +\infty$.

In this limit, $\epsilon_b(\omega)$ cannot be neglected & $\epsilon = i\frac{\sigma}{\omega}$ breaks down. The medium behaves as a regular insulator.