

HW #2 solutions

13.2 Equal and Opposite Magnetization

- (a) There is no free current. The magnetization in each region is uniform so the bulk magnetization current density $\mathbf{j}_M = \nabla \times \mathbf{M} = 0$. The magnetization is normal to the $z = 0$ interface so the surface magnetization current density $\mathbf{K} = \mathbf{M} \times \hat{\mathbf{n}} = 0$. There is no source current of any kind, so $\mathbf{B} = 0$ everywhere.
- (b) There is no bulk magnetic charge $\rho^* = -\nabla \cdot \mathbf{M}$ but there is a surface charge density $\sigma^* = \mathbf{M} \cdot \hat{\mathbf{n}}$. There is a contribution $\sigma = M$ at $z = 0$ due to the $z > 0$ region. An identical contribution comes from the $z < 0$ region. Therefore, since an outward-pointing electric field $E = \sigma/2\epsilon_0$ is created by a planar surface density of electric charge σ , we get an outward-pointing field $H = M$ in this case. Since \mathbf{M} points inward to the same interface, we conclude that $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}) = 0$ everywhere.

13.5 The Virtues of Magnetic Charge

- (a) The text establishes that $\mathbf{m} = \int d^3r \mathbf{M}$. On the other hand, using the proposed formula, the k^{th} component of the magnetic dipole moment of the sample is

$$m_k = - \int d^3r r_k \nabla \cdot \mathbf{M} = - \int d^3r \nabla \cdot (\mathbf{M} r_k) + \int d^3r (\mathbf{M} \cdot \nabla) r_k = \int d^3r M_k.$$

(b) By definition, the interaction energy between two current distributions is

$$\hat{V}_B = -\frac{\mu_0}{4\pi} \int d^3r \int d^3r' \frac{\mathbf{j}_1(\mathbf{r}) \cdot \mathbf{j}_2(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}.$$

Using the definition of the vector potential in the Coulomb gauge, this is

$$\hat{V}_B = - \int d^3r \mathbf{j}_1 \cdot \mathbf{A}_2 = - \int d^3r \mathbf{A}_2 \cdot \nabla \times \mathbf{M}_1 = \int d^3r \nabla \cdot (\mathbf{A}_2 \times \mathbf{M}_1) - \int d^3r \mathbf{M}_1 \cdot \nabla \times \mathbf{A}_2.$$

Finally, using the divergence theorem and the fact that \mathbf{M}_1 is zero on the integration surface at infinity, we conclude that

$$\hat{V}_B = - \int d^3r \mathbf{M}_1 \cdot \mathbf{B}_2.$$

Precisely the same steps beginning with $\hat{V}_B = - \int d^3r \mathbf{j}_2 \cdot \mathbf{A}_1$ establish the reciprocity relation.

(c) It is simplest to begin with the proposed formula and show that it is equivalent to the expression derived in part (b). Then, because $\mathbf{B}_2 = \mu_0 \mathbf{H}_2$ in the part of space where $\mathbf{M}_1 \neq 0$,

$$\begin{aligned} \hat{V}_B &= \frac{\mu_0}{4\pi} \int d^3r \int d^3r' \frac{\nabla \cdot \mathbf{M}_1(\mathbf{r}) \nabla' \cdot \mathbf{M}_2(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \\ &= \frac{\mu_0}{4\pi} \int d^3r' \nabla' \cdot \mathbf{M}_2(\mathbf{r}') \int d^3r \left\{ \nabla \cdot \left[\frac{\mathbf{M}_1(\mathbf{r})}{|\mathbf{r} - \mathbf{r}'|} \right] - \mathbf{M}_1(\mathbf{r}) \cdot \nabla \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right\} \\ &= \int d^3r \mathbf{M}_1(\mathbf{r}) \cdot \nabla \frac{\mu_0}{4\pi} \int d^3r' \frac{\rho_2^*(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \\ &= - \int d^3r \mathbf{M}_1(\mathbf{r}) \cdot \mu_0 \mathbf{H}_2(\mathbf{r}) \\ &= - \int d^3r \mathbf{M}_1(\mathbf{r}) \cdot \mathbf{B}_2(\mathbf{r}). \end{aligned}$$

13.11 Lunar Magnetism

We have $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$, where $\mathbf{H} = -\nabla\psi$ and ψ satisfies the Poisson-like equation

$$\nabla^2\psi = \nabla \cdot \mathbf{M}.$$

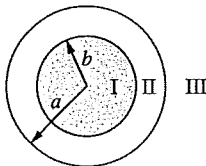
In addition, at the boundary between regions, it is necessary to satisfy the matching conditions

$$\psi_1(\mathbf{r}_S) = \psi_2(\mathbf{r}_S)$$

and

$$\left[\frac{\partial\psi_1}{\partial n_1} - \frac{\partial\psi_2}{\partial n_1} \right]_S = [\mathbf{M}_1 - \mathbf{M}_2]_S \cdot \hat{\mathbf{n}}_1.$$

We call the core, crust, and exterior of the Moon regions I, II, and III, respectively, as shown below.



The impressed magnetization \mathbf{M} of the core is stated to be proportional to a dipole field \mathbf{B}_d centered at the origin. If we align the magnetic moment \mathbf{m} with the z -axis,

$$\mathbf{B}_d(r, \theta) = \frac{\mu_0 m}{4\pi} \frac{3 \cos \theta \hat{\mathbf{r}} - \hat{\mathbf{z}}}{r^3} = \frac{\mu_0 m}{4\pi} \frac{2 \cos \theta \hat{\mathbf{r}} + \sin \theta \hat{\boldsymbol{\theta}}}{r^3}.$$

Since $\nabla \cdot \mathbf{B} = 0$, we know that $\nabla \cdot \mathbf{M} = 0$ and the magnetic scalar potential above satisfies Laplace's equation everywhere. Specifically,

$$\psi_I = D \left(\frac{r}{b} \right) \cos \theta$$

$$\psi_{II} = \left[B \left(\frac{a}{r} \right)^2 + C \left(\frac{r}{a} \right) \right] \cos \theta$$

$$\psi_{III} = A \left(\frac{a}{r} \right)^2 \cos \theta.$$

Applying the matching conditions, noting that $\hat{\mathbf{n}} = \hat{\mathbf{r}}$ and that the only non-zero magnetization is

$$\mathbf{M}_{II} = M \frac{2 \cos \theta \hat{\mathbf{r}} + \sin \theta \hat{\boldsymbol{\theta}}}{r^3},$$

gives

$$A = B + C$$

$$D = B \frac{a^2}{b^2} + C \frac{b}{a}$$

$$\frac{2M}{a^3} = -\frac{2B}{a} + \frac{C}{a} + \frac{2A}{a}$$

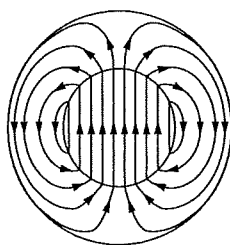
$$-\frac{2M}{b^3} = \frac{D}{b} + 2B \frac{a^2}{b^3} - \frac{C}{a}.$$

It is straightforward to check that this system is solved by

$$A = 0 \quad C = -B = \frac{2M}{3a^2} \quad D = B \left[\frac{a^2}{b^2} - \frac{b}{a} \right],$$



which confirms that $\mathbf{H} = \mathbf{B} = 0$ in region III outside the Moon. We can sketch \mathbf{B} inside the Moon using the fact that $\mathbf{B}_I = \mathbf{H}_I = -\nabla\psi_I$ is constant, the lines of \mathbf{B} must form closed loops, and \mathbf{B} must be tangent to the sphere at $r = b$ because its radial component is continuous there.



Source: S.K. Runcorn, *Physics of the Earth and Planetary Interiors* 10, 327 (1975).