<u>Solutions</u>

Final Exam:
E&M II 504

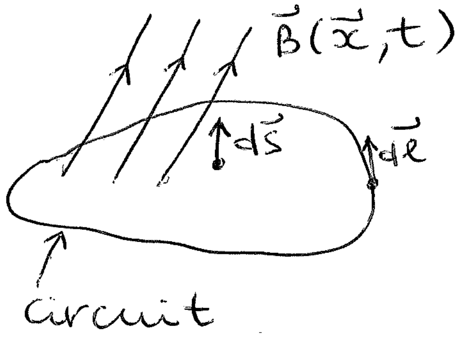
Spring 2018

Problem 1 [20 points]

Earth magnetism.
Rapid changes in the Earth's magnetic field during storms have been known to cause widespread power outages and surges in power lines. Let us try to investigate this phenomenon quantitatively.

1. Consider a conducting circuit in a magnetic field $\mathbf{B}(\mathbf{x}, t)$. Write down an expression for EMF around this circuit induced by the changes in $\mathbf{B}$.
2. Suppose now that the conducting circuit is a square of side l. Consider a spatially uniform $\mathbf{B}$ changing with time at a rate $\partial|\mathbf{B}|/\partial t$. Derive an expression for the EMF around the circuit if $\mathbf{B}$ is (a) perpendicular and (b) parallel to the circuit.
3. For the perpendicular case, suppose that $l = 100$ m and that $\partial|\mathbf{B}|/\partial t = 10^{-4}$ Gauss/s, a typical value during a storm. What is the induced EMF, in volts? if the circuit's resistance is 100 Ohm, what is the energy dissipated in such a circuit per unit time, in kilowatts?
4. Estimate the energy stored in the Earth's magnetic field by assuming for simplicity that the field is constant with $|\mathbf{B}| = 1$ Gauss and that it fills the volume of the Earth ($R = 6000$ km). Give the answer in kilowatt-hours (1 J $= 2.78 \times 10^{-7}$ kW-h, 1 erg $= 2.78 \times 10^{-14}$ kW-h). What is the fraction of the total available magnetic energy dissipated by a circuit from part 3, if the storm lasts 2 hours?

1.

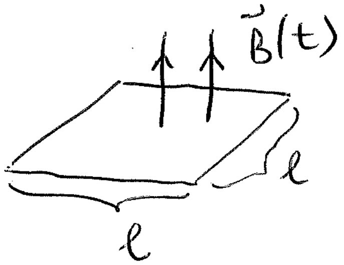


SI units

Use $\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$, or

$$\text{EMF} = \int_{\text{circuit}} \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\underbrace{\int \vec{B}\cdot d\vec{S}}_{\text{flux through the enclosed area}}$$

2.



(a) $\vec{B} \perp$ circuit:

$$\text{EMF} = l^2 \frac{\partial |\vec{B}|}{\partial t}$$

(b) $\vec{B} \parallel$ circuit:

$$\text{EMF} = 0$$

3. $\text{EMF} = 10^{-4}\frac{\text{gauss}}{\text{s}} \times \overset{10^{-4}\,\frac{\text{Tesla}}{\text{gauss}}}{}(10^{2}\,\text{cm})^2 = 10^{-4}$ Volts.

$\text{Power} = \frac{V^2}{R} = 10^{-10}$ W $= 10^{-13}$ kW.

EMF = V here

in gaussian units

4. $\text{Total}\ \overset{\text{magnetic}}{\text{energy}} \simeq \frac{4\pi}{3}R^3\frac{B^2}{8\pi} = \frac{1}{6}(6\times10^8\,\text{cm})^3 \times$

$\times (1\ \text{Gauss})^2 \simeq 3.6\times10^{25}$ ergs $\simeq 10^{12}$ kW-hours.

During the storm, 2×10^{-13} kW-hours is dissipated in the circuit, or 2×10^{-25} of the total magnetic energy.

Problem 2 [20 points]

Plane waves.

1. Consider two counter-propagating plane waves: $\mathbf{E}(z,t) = E_0\cos(kz-\omega t)\hat{x} + E_0\cos(kz+\omega t)\hat{x}$. Find the corresponding magnetic field $\mathbf{B}(z,t)$.
2. Find the time-averaged electric and magnetic field energy densities and the time-averaged Poynting vector.

1. Note that

$$\vec{E} = 2E_0\cos(kz)\cos(\omega t)\,\hat{x}$$, a standing wave

$$\vec{E} = \mathrm{Re}\,\vec{\mathcal{E}}$$, where

$$\vec{\mathcal{E}}(z,t) = \underbrace{\vec{E}'(z)}_{2E_0\cos(kz)\hat{x}}\, e^{-i\omega t}$$

Define $\vec{\mathcal{B}}(z,t) = \vec{B}'(z)\, e^{-i\omega t}$, then

$$\vec{\nabla}\times\vec{\mathcal{E}} = -\frac{\partial\vec{\mathcal{B}}}{\partial t} \Rightarrow \underbrace{\vec{\nabla}\times\vec{E}'(z)}_{-2kE_0\sin(kz)\hat{y}} = i\omega\vec{B}'(z).$$

$$\vec{\mathcal{B}} = \overbrace{\frac{2ik}{\omega}E_0\sin(kz)\,\hat{y}}^{\vec{B}'(z)}\, e^{-i\omega t}$$, or

$$\vec{B} = \mathrm{Re}\,\vec{\mathcal{B}} = \underline{\underline{\frac{2}{c}E_0\sin(kz)\sin(\omega t)\,\hat{y}}}.$$

2.

$$\underset{\substack{\uparrow \\ \text{time-averaged}}}{\langle} u_E(z) \rangle = \frac{1}{2}\,\frac{\epsilon_0}{2}\,\vec{E}'(z)^{*}\cdot \vec{E}'(z) =$$

$$= \epsilon_0 E_0^2 \cos^2(kz).$$

$$\langle u_B(z) \rangle = \frac{1}{2}\,\frac{\overbrace{\epsilon_0 c^2}^{\mu_0}}{2}\,\vec{B}'(z)^{*}\cdot \vec{B}'(z) =$$

$$= \underline{\underline{\epsilon_0 E_0^2 \sin^2(kz)}}.$$

Finally,

$\langle \vec{S} \rangle = \frac{1}{2}\,\mathrm{Re}\left(\vec{E}'(z)^{*} \times \vec{B}'(z)\right) = 0$, as expected from a standing wave.

Problem 3 [20 points]

EM wave scattering by a bound electron.
Consider an electron with mass m and charge $-e$ harmonically bound at the origin with natural frequency ω_0. The electron is immersed in an incoming plane wave with $\mathbf{k} = k\hat{x}$, frequency ω, and fields $E_0\hat{y}$ and $B_0\hat{z}$. Assume that the bound electron moves only along $\hat{y}$.

1. Calculate the time-dependent electric dipole moment $\mathbf{p}(t)$ for the bound electron.
2. Calculate the electric $\mathbf{E}$ and the magnetic $\mathbf{B}$ fields radiated by the electron in the radiation zone. What are their polarizations relative to $\mathbf{p}$?
3. Calculate the time-averaged power per unit solid angle, $dP/d\Omega$, and the total power, P, radiated by the electron. Show your work!
4. Sketch the radiation intensity pattern and explain why this radiation is called "electric dipole radiation".

1. The EoM is

$$\vec{F} = m\vec{a} \quad , \text{ where } \quad \vec{F} = \vec{F}_E + \vec{F}_{spring} :$$

$$\vec{F} = \left(-eE_0 \cos(\omega t) - m\omega_0^2 y\right)\hat{y} .$$

Thus $$m\frac{d^2y}{dt^2} = -eE_0\cos(\omega t) - m\omega_0^2 y ,$$

which is solved by

$$y(t) = \frac{eE_0}{m(\omega^2-\omega_0^2)}\cos(\omega t) .$$

Finally, $$\vec{p}(t) = -e\vec{r}(t) = -\frac{e^2E_0}{m(\omega^2-\omega_0^2)}\cos(\omega t)\hat{y} .$$

(*)

2. The rest is standard (see Jackson & lecture notes for details):

In the radiation zone,

$$\begin{cases} \vec{B} = \dfrac{Z_0 k^2}{4\pi} (\vec{n} \times \vec{p}) \dfrac{e^{ikr}}{r}, \\ \vec{E} = \dfrac{1}{c} \vec{B} \times \vec{n}, \quad \text{where } \vec{p} \text{ is given by } (*) \end{cases}$$

$\vec{B} \perp \vec{p}, \vec{n}$ & $\vec{E} \perp \vec{B}, \vec{n}$

time-averaged over oscillations

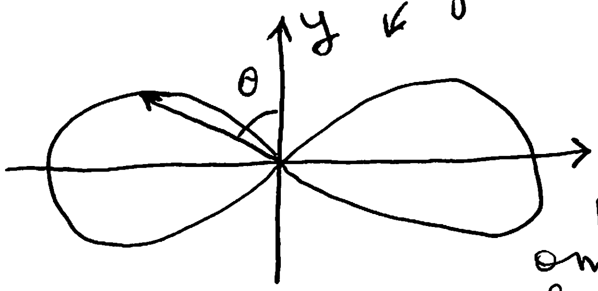
3. $$\frac{dP}{d\Omega} = \frac{c^2 Z_0}{32\pi^2} k^4 \underbrace{|\vec{p}|^2}_{\text{here,}} \sin^2\theta$$

$$|\vec{p}|^2 = \frac{e^4 E_0^2}{m^2(\omega^2 - \omega_0^2)^2},$$ since $\cos(\omega t)$ has already been averaged over.

Finally, $$P = \frac{c^2 Z_0 k^4}{12\pi} |\vec{p}|^2$$

total power

dipole oscillates along y axis

4. 

The radiation pattern is called an "electric dipole radiation pattern" since it is omitted by an oscillating electric dipole, and since it looks like $\vec{E}$ for a static electric dipole.

Problem 4 [20 points]

Lorentz Transformations.

1. Under an infinitesimal Lorentz boost along the z-axis, the coordinates are transformed as

$$x_3' = x_3 - \delta\lambda x_0, \quad x_0' = x_0 - \delta\lambda x_3$$

(x_1 and x_2 remain unchanged). Show that the light cone coordinates, $x_\pm = x_0 \pm x_3$, transform as

$$x_\pm' = e^{\mp\lambda} x_\pm$$

under a finite Lorentz transform, which consists of repeating the infinitesimal Lorentz boost $N \to \infty$ times: $\lambda = N\delta\lambda$, λ finite.

2. Find v, the velocity of the new frame as measured in the old frame, in terms of λ, and use it to write out explicit equations for the finite Lorentz transform.

1. For an infinitesimal boost,

$$x_\pm' = x_0' \pm x_3' = (x_0 - \delta\lambda x_3) \pm (x_3 - \delta\lambda x_0) =$$

$$= (1 \mp \delta\lambda) x_\pm \simeq e^{\mp\delta\lambda} x_\pm .$$

A finite boost is then given by

$$x_\pm' \underset{\lim\limits_{N\to\infty}}{=} (e^{\mp\frac{\lambda}{N}})^N = e^{\mp\lambda} \underline{\underline{x_\pm}} .$$

In other words,

$$\begin{cases} x_0' + x_3' = e^{-\lambda}(x_0 + x_3), \\ x_0' - x_3' = e^{\lambda}(x_0 - x_3). \end{cases}$$

2. From part 1,

$$\begin{cases} x_0' = x_0 \cosh\lambda - x_3 \sinh\lambda, \\ x_3' = x_3 \cosh\lambda - x_0 \sinh\lambda. \end{cases}$$

$x_3' = 0$ is the spatial origin in the new frame, yielding

$$x_3 = x_0 \frac{\sinh\lambda}{\cosh\lambda} = t\underbrace{c\frac{\sinh\lambda}{\cosh\lambda}}_{v}$$

in the old frame.

Thus $\frac{v}{c} = \frac{\sinh\lambda}{\cosh\lambda} \Rightarrow \left(\frac{v}{c}\right)^2 = 1 - \frac{1}{\cosh^2\lambda}$,

$$\cosh\lambda = \frac{1}{\sqrt{1 - v^2/c^2}} \Rightarrow \sinh\lambda = \frac{v/c}{\sqrt{1-v^2/c^2}}.$$

Finally,
$$\begin{cases} x_3' = \dfrac{x_3 - \frac{v}{c}x_0}{\sqrt{1-v^2/c^2}}, \\ x_0' = \dfrac{x_0 - \frac{v}{c}x_3}{\sqrt{1-v^2/c^2}} \end{cases}$$

↖ standard result

Problem 5 [20 points]

Covariance and stress-energy tensor.

1. Express $\mathbf{E}^2 - \mathbf{B}^2$ and $\mathbf{E} \cdot \mathbf{B}$ in terms of $F^{\mu\nu}$ and $\mathcal{F}^{\mu\nu}$, and thereby prove that they are Lorentz invariant.
2. Suppose there is an inertial frame in which the electromagnetic field is purely a magnetic field. Does another inertial frame exist in which the electromagnetic field is purely an electric field? Explain your reasoning.
3. Use the definition of the symmetric stress-energy tensor $\Theta^{\alpha\beta}$ in Jackson to show that

$$\partial_\alpha \Theta^{\alpha\beta} = -\frac{1}{c} F^{\beta\lambda} J_\lambda$$

Show your work!

1. $F^{\alpha\beta} F_{\alpha\beta} = 2(\vec{B}^2 - \vec{E}^2)$

$\mathcal{F}^{\alpha\beta} \mathcal{F}_{\alpha\beta} = -2(\vec{B}^2 - \vec{E}^2)$

$\mathcal{F}^{\alpha\beta} F_{\alpha\beta} = -4(\vec{E} \cdot \vec{B})$

Thus $\vec{E}^2 - \vec{B}^2$ and $\vec{E} \cdot \vec{B}$ are Lorentz inv.

2. No, since if $\vec{E} = 0$ & $\vec{B} \neq 0$, in some frame, $\vec{E}^2 - \vec{B}^2 < 0$ in any frame.

$\vec{E} \neq 0$, $\vec{B} = 0$ gives $\vec{E}^2 - \vec{B}^2 > 0$.

3. $\Theta^{\alpha\beta} = \frac{1}{4\pi}\left(g^{\alpha\mu} F_{\mu\lambda} F^{\lambda\beta} + \frac{1}{4} g^{\alpha\beta} F_{\mu\lambda} F^{\mu\lambda}\right).$

Then $\partial_\alpha \Theta^{\alpha\beta} = \frac{1}{4\pi}\left[\partial^\mu (F_{\mu\lambda} F^{\lambda\beta}) + \right.$

$\left. + \frac{1}{4}\partial^\beta (F_{\mu\lambda} F^{\mu\lambda})\right] =$

$$= \frac{1}{4\pi}\left[\underbrace{(\partial^\mu F_{\mu\lambda})}_{\frac{4\pi}{c} J_\lambda} F^{\lambda\beta} + F_{\mu\lambda}(\partial^\mu F^{\lambda\beta}) + \underbrace{\frac{1}{4}(\partial^\beta F_{\mu\lambda}) F^{\mu\lambda} + \frac{1}{4} F_{\mu\lambda}(\partial^\beta F^{\mu\lambda})}_{\frac{1}{2} F_{\mu\lambda}(\partial^\beta F^{\mu\lambda})}\right],$$

or

$$\partial_\alpha \Theta^{\alpha\beta} + \frac{1}{c} F^{\beta\lambda} J_\lambda = \frac{1}{8\pi}\left[\partial^\mu F^{\lambda\beta} + \underbrace{\partial^\mu F^{\lambda\beta} + \partial^\beta F^{\mu\lambda}}_{\partial^\lambda F^{\mu\beta}}\right] F_{\mu\lambda} =$$

$$= \frac{1}{8\pi}\underbrace{F_{\mu\lambda}}_{\substack{\text{antisym.}\\ \text{in } \mu,\lambda}}\left[\underbrace{\partial^\mu F^{\lambda\beta} + \partial^\lambda F^{\mu\beta}}_{\text{sym. in } \mu,\lambda}\right] = 0,$$ so that

$$\underline{\underline{\partial_\alpha \Theta^{\alpha\beta} = -\frac{1}{c} F^{\beta\lambda} J_\lambda.}}$$