Homework 5 Solutions

Physics 406

1. O chapter 4, Q 8

$R_H \sim 1/N$

The Hall constant R_H is defined as

$$R_H \equiv \frac{E_H}{J_x B}$$

where E_H is the Hall field, B is the applied field and J_x is an electric current. Current is charge / unit and thus is proportional to the concentration of electrons $N \Rightarrow R_H \sim 1/N$.

2. O chapter 4, Q 9

Positive charge carriers will mean $J = N(+e) v_x$

$$\Downarrow$$

$$R_H = \frac{E_H}{J_x B} \sim \frac{1}{+e} > 0.$$

3. O chap 5 Q 1.

Here we must distinguish between core (localized) and valence (delocalized) electrons. The combination of the core electrons and the ions leads to a weak pseudopotential; the core electrons thus "screen" the ions. Bloch's theorem tells us that the periodic nature of this resulting potential results in valence electrons with propagating wavefunctions. Thus there is no paradox - rather there are two types of electrons.

4. ◊ chap. 4 P 10

$$\nu_c = \frac{\omega_c}{2\pi} = 2.8 \; B \; GHz \quad \text{for} \quad m = m^*$$

$$\nu_c = 24 \; GHz \Rightarrow B = \frac{24}{2.8} \; k\,Gauss$$

$$\Downarrow$$

$$\boxed{B = 8.6 \text{ kilogauss}.}$$

5. Kittel chapter 6 P 6

$$m\left(\frac{dv}{dt} + \frac{v}{\tau}\right) = -eE$$

Let $v = v_0 e^{-i\omega t}$

$E = E_0 e^{-i\omega t}$

$$\Downarrow$$

$$\left(-i\omega + 1/\tau\right) v_0 = -eE_0/m$$

$$\Downarrow$$

$$v_0 = \frac{-eE_0/m}{-i\omega + 1/\tau} = \frac{-eE}{m}\,\frac{1+i\omega\tau}{1+(\tau\omega)^2}$$

$$j = n(-e)v = \frac{ne^2\tau}{m}\,\frac{1+i\omega\tau}{1+(\tau\omega)^2}\,E$$

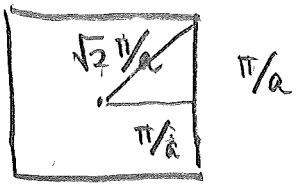
$$\equiv \sigma E$$

$$\Downarrow$$

$$\boxed{\sigma(\omega) = \sigma(0)\,\frac{1+i\omega\tau}{1+(\tau\omega)^2}}$$

6. Kittel chap. 7 P 1.

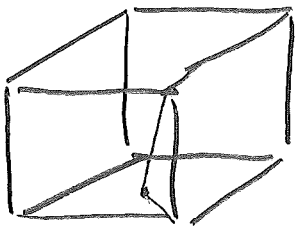
a)



$$k_{inner} = \sqrt{2}\, k_{center}$$

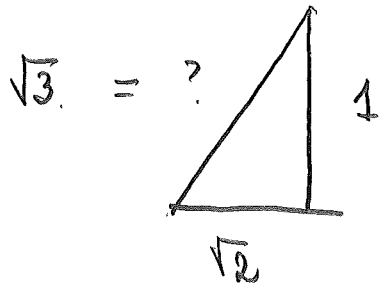
$$E \sim k^2 \Rightarrow \frac{E_{inner}}{E_{center}} = 2$$

b)



$$k_{inner} = \sqrt{3}\, k_{center}$$

$$E \sim k^2 \Rightarrow \frac{E_{inner}}{E_{center}} = 3$$



c) If $E_{gap} < E_{inner} - E_{center}$

electrons will find it energetically more favorable to occupy center states in the 2nd zone rather

than filling up inner states in the 1st Brillouin zone. Under these conditions divalent elements could be metals and not insulators.

7. Kittel p. 6.

The central equation is

$$(\lambda_k - \varepsilon)\, c_k + \sum_G U_G\, c_{k-G} = 0$$

We need to find U_G

$$U(x) = \sum_G U_G\, e^{iGx}$$

$$\Downarrow$$

$$U_G = \frac{1}{a^2} \int_{cell} U(x)\, e^{-i\vec{G}\cdot\vec{x}}\, dx\, dy$$

$$\vec{G} = \left(\frac{2\pi}{a}, \frac{2\pi}{a} \right)$$

Let $x_i = \begin{cases} x & i = 1 \\ y & i = 2. \end{cases}$

Then

$$u_G = -\frac{4u}{a^2} \prod_{i=1}^{2} \int_0^a dx_i \cos\frac{2\pi x_i}{a} \, e^{-i\frac{2\pi x_i}{a}}$$

$$= -\frac{4u}{a^2} \prod_{i=1}^{2} \int_0^a \frac{dx_i}{2} \left[e^{\frac{i2\pi x_i}{a}} + e^{-i\frac{2\pi x_i}{a}} \right] \times e^{-i\frac{2\pi x_i}{a}}$$

$$= -\frac{4u}{a^2} \frac{1}{4} \prod_{i=1}^{2} \int_0^a dx_i \left[1 + e^{-i\frac{4\pi x_i}{a}} \right]$$

integral = 0

since $e^{2\pi i n} = 1$

$$= -\frac{u}{a^2} [a][a]$$

$$= -u \quad \Rightarrow \quad u_G = -u$$

Then the central equations can be written in matrix form

$$\begin{bmatrix} \lambda_k - \varepsilon & -u \\ -u & \lambda_{k-G} - \varepsilon \end{bmatrix} \begin{bmatrix} c_k \\ c_{k-G} \end{bmatrix} = 0 .$$

$$\det [\ ] = 0 \Rightarrow (\lambda_k - \varepsilon)(\lambda_{k-G} - \varepsilon) - u^2 = 0$$

$$\lambda_k = \frac{\hbar^2 k^2}{2m}$$

$$\vec{k} = \left(\frac{\pi}{a}, \frac{\pi}{a} \right) \Rightarrow \vec{k} - \vec{G} = \left(-\frac{\pi}{a}, -\frac{\pi}{a} \right)$$

$$\vec{G} = \left(\frac{2\pi}{a}, \frac{2\pi}{a} \right)$$

$$\Downarrow$$

$$\lambda_k = \lambda_{k-G}$$

Therefore the equations we need to solve is

$$(\lambda_k - \varepsilon)^2 = u^2$$

$$\Downarrow$$

$$\varepsilon_\pm = \lambda \pm u$$

$$\Downarrow$$

$$\text{gap} = \varepsilon_+ - \varepsilon_- = 2u.$$

8. O chapter 5 P 12.

a) 1d $\quad k = \left(\frac{2\pi}{L}\right) n$

$$\Downarrow$$

$$N = \frac{kL}{2\pi} \quad \Rightarrow \quad \frac{dN}{dk} = \frac{L}{2\pi}$$

$$g(E) = \frac{dN}{dE} = \frac{dN}{dk}\,\frac{dk}{dE}$$

$$g(E) = \frac{\frac{L}{2\pi}}{\frac{dE}{dk}}$$

b) For 1d the tight-binding model yields

$$E(k) = E_0 + 4\gamma \sin^2 \frac{ka}{2}.$$

$$\frac{dE}{dk} = 4\gamma\, 2 \sin\frac{ka}{2}\left(\frac{a}{2}\right) \cos\frac{ka}{2}.$$

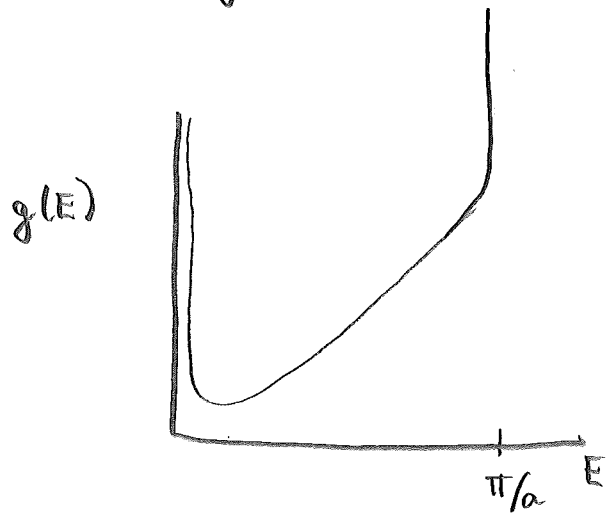
$$\frac{dE}{dk} = 4\gamma a \sin\frac{ka}{2} \cos\frac{ka}{2}$$

$$\Downarrow$$

$$g(E) = \frac{L/2\pi}{4\gamma a \sin\frac{ka}{2}\cos\frac{ka}{2}}$$

Limits

- $k \sim 0 \qquad g(E) \sim 1/k \sim E^{-1/2}$
- $k \sim \pi/a \qquad g(E) \to \infty$



9. O chapter 5 P 14

a) Free Electron Model

$$n = 2 \frac{1}{(2\pi)^3} \frac{4}{3} \pi k_F^3$$

$$\Downarrow$$

$$n = \frac{1}{3\pi^2} k_F^3 \Rightarrow k_F = (3\pi^2 n)^{1/3}$$

b) Fermi sphere touches face of 1st BZ of fcc lattice

when

$$k_F = k_i$$

Here k_i is the shortest distance from the origin to the 1BZ boundary $\Rightarrow$

$$k_i = \frac{1}{2} a_i^* \quad \text{for fcc}$$

Reciprocal lattice vectors for fcc lattice

$$a^* = \frac{2\pi}{a}(1,-1,1) \qquad c^* = \frac{2\pi}{a}(-1,1,1)$$

$$b^* = 2\pi/a\ (1,1,-1)$$

Then

$$k_i = \frac{1}{2}\ \frac{2\pi}{a}\ (1^2+1^2+1^2)^{1/2} = \frac{\sqrt{3}\,\pi}{a}.$$

For fcc lattice

$$n = \frac{4 \text{ atoms}}{a^3} \times \frac{n}{n_a}$$

$$k_F = k_i$$

$$\left(3\pi^2\ \frac{4}{a^3}\ \frac{n}{n_a}\right)^{1/3} = \sqrt{3}\ \frac{\pi}{a}$$

$$\left(\frac{12}{\pi}\right)^{1/3}\ \left(\frac{n}{n_a}\right)^{1/3} = \sqrt{3}$$

$$\Downarrow$$

$$\frac{n}{n_a} = \sqrt{3}\ \frac{3}{12}\ \frac{\pi}{4}$$

$$\frac{n}{n_a} = \frac{\sqrt{3}\,\pi}{4} = 1.36.$$

e) Zn divalent
Cu monovalent

$$n = \frac{4}{a^3} \{ (1-x)(1) + x(2) \}$$

↓ concentration of Zn

$$n = \frac{4}{a^3} \{ 1 + x \}$$

$k_F = k_i$ for FS to touch face of 1st BZ

$$(3\pi^2 n)^{1/3} = \sqrt{3}\,\frac{\pi}{a}$$

$$\left(\frac{12}{\pi}\right)^{1/3} (1+x)^{1/3} = \sqrt{3}$$

$$(1+x) = \frac{\pi}{12}\, 3\sqrt{3}$$

$$x = \frac{\pi\sqrt{3}}{4} - 1 = .36$$

Concentration of Zn = 36%

Concentration of Cu = 64%.

Atomic ratio

$$X = \frac{x_{Zn}}{A_{Zn}} = \frac{36}{65.4\ g/mol} = .55$$

$$y = \frac{(64)}{A_{Cu}} = \frac{64}{63.5\ g/mol} = 1.007$$

$Cu_{1.007}\ Zn_{.55}$

⇓ Divide each by .55

$$\boxed{Zn_1\ Cu_{1.832}.}$$

Zn concentration = 36 % = C_{Zn}

Cu concentration = 64 % = C_{Cu}

Atomic ratios

$$X = \frac{C_{Zn}}{A_{Zn}} = \frac{36}{65.4} = .55$$

$$y = \frac{C_{Cu}}{A_{Cu}} = \frac{64}{63.5} = 1.007$$

We can therefore write this alloy as

$Zn_{.55}\ Cu_{1.007}$

⇓ or as (dividing by .55)

$Zn_1\ Cu_{1.832}$

⇓

$Zn_5\ Cu_9$

(since formula must have integer # of constituent atoms)

10. Developments in the field of photonic crystals (based on Handout 6)

Idea: Develop periodic structures of materials w/ suitable refractive indices such that incoming light of a particular wavelength results in standing waves due to reflection and refraction

Hope: Analog of doping in semiconductors can be achieved by slightly modifying structure (e.g. extra holes)

Challenges:
- each structural unit must produce reflected and refracted waves that cancel out incoming waves travelling in all directions
- Identification of appropriate structures and materials with necessary refractive indices

Impact:
- Nanolasers
- High capacity optical fibers