Solution 6

Physics 313

7.23 $\text{Prob}_{2,1,1} = \int_{x=0,y=L/3,z=0}^{x=L,y=2L/3,z=L} \left(A\sin\frac{2\pi x}{L}\sin\frac{1\pi y}{L}\sin\frac{1\pi z}{L}\right)^2 dxdydz$. Using the fact that $A = (2/L)^{3/2}$, this becomes

$$\text{Prob}_{2,1,1} = \left(\frac{2}{L}\right)^3\left(\int_0^L \sin^2\frac{2\pi x}{L}dx\right)\left(\int_{L/3}^{2L/3}\sin^2\frac{l\pi y}{L}dy\right)\left(\int_0^L \sin^2\frac{l\pi z}{L}dz\right)$$

But $\int_{w_1}^{w_2}\sin^2\frac{n\pi w}{L}dw = \int_{w_1}^{w_2}\left(\frac{1}{2}-\frac{1}{2}\cos\frac{2n\pi w}{L}\right)dw = \frac{w_2-w_1}{2} - \frac{L}{4n\pi}\left(\sin\frac{2n\pi w_2}{L}-\sin\frac{2n\pi w_1}{L}\right)$. Thus,

$$\text{Prob}_{2,1,1} = \left(\frac{2}{L}\right)^3\left(\frac{L}{2}\right)^2\left(\frac{2L/3-L/3}{2}-\frac{L}{4\pi}\left(\sin\frac{2\pi 2L/3}{L}-\sin\frac{2\pi L/3}{L}\right)\right) = \frac{2}{L}\left(\frac{L}{6}+\frac{L}{4\pi}\sqrt{3}\right) = \frac{1}{3}+\frac{\sqrt{3}}{2\pi} = \mathbf{0.609}$$

(b) $\text{Prob}_{1,2,1} = \int_{x=0,y=L/3,z=0}^{x=L,y=2L/3,z=L}\left(A\sin\frac{1\pi x}{L}\sin\frac{2\pi y}{L}\sin\frac{1\pi z}{L}\right)^2 dxdydz$

$$= \left(\frac{2}{L}\right)^3\left(\int_0^L \sin^2\frac{1\pi x}{L}dx\right)\left(\int_{L/3}^{2L/3}\sin^2\frac{2\pi y}{L}dy\right)\left(\int_0^L \sin^2\frac{1\pi z}{L}dz\right)$$

$$= \left(\frac{2}{L}\right)^3\left(\frac{L}{2}\right)^2\left(\frac{2L/3-L/3}{2}-\frac{L}{8\pi}\left(\sin\frac{4\pi 2L/3}{L}-\sin\frac{4\pi L/3}{L}\right)\right)$$

$$= \frac{2}{L}\left(\frac{L}{6}-\frac{L}{8\pi}\sqrt{3}\right) = \frac{1}{3}-\frac{\sqrt{3}}{4\pi} = \mathbf{0.196}$$

(c) Because the limits of integration of x and z are the same, $\text{Prob}_{1,1,2}$ must equal $\text{Prob}_{2,1,1}$ = **0.609**. The region includes one third of the well. The probability is less than a third for (1,2,1) because this center slice along the y–axis is centered on a node in the standing wave. The other two probabilities are large because the slice is centered on an antinode.

7.28 The longest wavelength Lyman series (ending on n = 1) line starts at n = 2. From Figure 7.5 we see that the energy difference is 10.2eV, and Example 7.2 shows the wavelength of this line to be 122nm, far shorter than visible. The shortest wavelength Paschen series (ending on n = 3) line starts at the largest n possible, giving an energy difference of 1.5eV. $E = h\frac{c}{\lambda} \rightarrow 1.5\text{eV} = \frac{1240\text{eV}\cdot\text{nm}}{\lambda} \Rightarrow \lambda = 827\text{nm}$ far longer than visible. We know that the first four Balmer lines are visible. What of that for which $n_i = 7$? $E_{\text{electron, initial}} - E_{\text{electron,final}} = \frac{-13.6\text{eV}}{7^2} - \frac{-13.6\text{eV}}{2^2} = 3.12\text{eV}$. $3.12\text{eV} = \frac{1240\text{eV}\cdot\text{nm}}{\lambda} \Rightarrow \lambda = 397\text{nm}$. This is slightly shorter than the usually quoted visible range, and any higher-energy lines in the series would have even shorter wavelengths.

7.32 $\Delta E = E_4 - E_1 = (-13.6\text{eV})\left(\frac{1}{4^2}-\frac{1}{2^2}\right) = 2.55\text{eV}.$ $E = h\frac{c}{\lambda} \rightarrow 2.55\text{eV} = \frac{1240\text{eV}\cdot\text{nm}}{\lambda} \Rightarrow \lambda = \mathbf{486nm}.$

(b) The electron could jump back down to n = 2, emitting a **486nm** photon, then to the n = 1 (2→1), or it could jump to the n = 3 (4→3) then to the n = 1 (3→1), or n = 2 (3→2) then n = 1. It might also jump direct down to the n = 1. As calculated in Example 7.2, the wavelengths of the 3→1, the 3→2, and the 2→1 are **103nm, 656nm** and **122nm**. For the 4→3 and 4→1 we may use Equation (7-13): $\frac{1}{\lambda} = 1.097\times 10^7\text{m}^{-1}\left(\frac{1}{3^2}-\frac{1}{4^2}\right) \Rightarrow \lambda = \mathbf{1875nm}$; $\frac{1}{\lambda} = 1.097\times 10^7\text{m}^{-1}\left(\frac{1}{1^2}-\frac{1}{4^2}\right) \Rightarrow \lambda = \mathbf{97.2nm}.$

7.36 Setting the function equal at $\phi = 0$ and $\phi = 2\pi$, $e^{i\sqrt{D}2\pi} = e^{i\sqrt{D}0} = 1 \rightarrow \cos(\sqrt{D}2\pi)+i\sin(\sqrt{D}2\pi) = 1$. If the real part is to be 1, $\sqrt{D}$ must be an integer, at which values the imaginary part is zero.

7.45 $\frac{-13.6\text{eV}}{4^2} = \mathbf{-0.85eV}$.

(b) **Magnitude of angular momentum**: $L = \sqrt{\ell(\ell+1)}\,\hbar$ where $\ell = 0, 1, \ldots, n-1$. Thus ℓ can be 0, 1, 2, 3, with $L = \mathbf{0}, \boldsymbol{\sqrt{2}\,\hbar}, \boldsymbol{\sqrt{6}\,\hbar}, \boldsymbol{\sqrt{12}\,\hbar}$. **Z–component of angular momentum**: $L_z = m_\ell \hbar$ where $m_\ell = -\ell, -\ell+1, \ldots, -1, 0, +1, \ldots, \ell-1, \ell$. With ℓ as large as 3, m_ℓ values could cover −3, −2, −1, 0, +1, +2, +3 with corresponding $L_z = \mathbf{-3\hbar, -2\hbar, -\hbar, 0, +\hbar, +2\hbar, +3\hbar}$.

7.46 For each ℓ, m_ℓ may take on values from $-\ell$ to $+\ell$ in integral steps; there are $2\ell+1$ such values. We must sum these values for all allowed values of ℓ: from zero to $n-1$ in integral steps.

$$\text{Total \#} = \sum_0^{n-1} 2\ell+1 = 2\sum_0^{n-1}\ell + \sum_0^{n-1} 1 = 2\,\frac{n(n-1)}{2} + n = n^2$$

7.47 $\int_0^\pi \Theta(\theta)^2 2\pi \sin\theta\, d\theta$ should be equal to unity, where $\Theta_{1,0}(\theta)$ is given by $\sqrt{\frac{3}{4\pi}}\cos\theta$.

$$\int_0^\pi \left(\sqrt{\frac{3}{4\pi}}\cos\theta\right)^2 2\pi\sin\theta\, d\theta = \frac{3}{2}\int_0^\pi \cos^2\theta \sin\theta\, d\theta = \frac{3}{2}\frac{-\cos^3\theta}{3}\bigg|_0^\pi = \frac{1}{2}(-\cos^3\pi + 1) = 1.\ \text{OK}$$