Solution to Problem Set 2

A system of fermions in the presence of impurities with concentration $n_i = N/V$ is described by the Hamiltonian,

$$\mathcal{H} = \sum_k \epsilon_k a_k^\dagger a_k + \int dx U(x)\psi^\dagger(x)\psi(x) \tag{1}$$

where $U(x) = \sum_i V(x - X_i)$ is the array of potentials due to impurities located at random positions X_i. The Fourier transform of the field operator is,

$$\psi(x) = \frac{1}{\sqrt{V}} \sum_k e^{ikx} a_k$$

and that of the impurity potential $V(x)$ is,

$$V(q) = \int dx e^{-iqx} V(x)$$

The Hamiltonian can be rewritten as

$$\mathcal{H} = \sum_k \epsilon_k a_k^\dagger a_k + \frac{1}{V} \sum_q V(q)\rho_q \sum_p a_{p+q}^\dagger a_p \tag{2}$$

where $\rho_q = \sum_i e^{-iqX_i}$, is a function of the random variables X_i. We have to perform ensemble average over the random positions. This makes the Hamiltonian translationally invariant and therefore tractable. For a typical sample the ensemble average is justified only if the probability distribution over the ensembles is sharply peaked. This condition is satisfied in the limit of weak disorder. The simplest ensemble average (with uniform weight) is defined as,

$$\overline{F} = \prod_{i=1}^{N} \frac{1}{V} \int d^3X_i F(X_1, \ldots, X_N) \tag{3}$$

Then,

$$\overline{\rho_q} = \frac{N}{V} \int d^3X e^{-iqX} = N\delta_{q,0} \tag{4}$$

This average will appear in the first order self energy diagram. For the second order term we have,

$$\overline{\rho_{q_1}\rho_{q_2}} = N^2\delta_{q_1,0}\delta_{q_2,0} + N\delta_{q_1+q_2,0}$$

where we have approximated $N(N-1)$ by N^2, assuming N to be large. For the calculation of self energy we need,

$$\overline{\rho_q\rho_{-q}} = N^2\delta_{q,0} + N \tag{5}$$

The first term on the right side produce second order improper self energy diagram, which can be disregarded. The second term will be relevant for second order proper self energy diagram.

In the Keldysh formalism Green's function G and self energy Σ are matrices defined as,

$$G = \begin{bmatrix} G^t & G^< \\ G^> & G^{\tilde{t}} \end{bmatrix}, \quad and \quad \Sigma = \begin{bmatrix} \Sigma^t & \Sigma^< \\ \Sigma^> & \Sigma^{\tilde{t}} \end{bmatrix}$$

Dyson's equation in frequency-momentum space takes the form,

$$G = G_0 + G_0\sigma_3\Sigma\sigma_3 G \tag{6}$$

where σ_3 is the Pauli matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Defining $\tilde{G} = G\sigma_3$, we can rewrite the above equation as

$$\tilde{G} = \tilde{G}_0 + \tilde{G}_0\tilde{\Sigma}\tilde{G} \tag{7}$$

So, the Feynmn rules to evaluate the diagrams are best written in terms of $\tilde{G}$ and $\tilde{\Sigma}$. Thus, a single line denotes the free propagator $i\tilde{G}_0$, the scattering vertex represents $\frac{-iV(q)\rho_q}{V}$, which has to be averaged according to equation (3). The self energy diagram gives $-i\tilde{\Sigma}$. Then, the first order self energy diagram, after the impurity average, yields,

$$\begin{aligned} -i\tilde{\Sigma}(k) &= -i\frac{N}{V}V(q=0) \\ \Sigma^t(k) &= n_i\int d^3xV(x) \\ \Sigma^{\tilde{t}}(k) &= -n_i\int d^3xV(x) \end{aligned} \tag{8}$$

To first order there is no contribution to $\Sigma^<$ and $\Sigma^>$. Then, $\Sigma^A = \Sigma^t - \Sigma^> = \Sigma^t$, and $\Sigma^R = \Sigma^t - \Sigma^< = \Sigma^t$. Also $\Sigma^K = \Sigma^< + \Sigma^> = 0$. Also to be noted is that the first order contribution to self energy is constant, and it leads to a simple shift in the energy levels of the free theory. Therefore, this contribution is of no physical interest.

The second order proper self energy diagram, after impurity agerage, gives,

$$-i\tilde{\Sigma}(k) = \frac{N}{V^2}\sum_p(-iV(p-k))(-iV(k-p))(i\tilde{G}(p)) \tag{9}$$

To get a closed form expression let us assume a short range potential, namely, $V(x) = V_0\delta(x)$. Since,

$$\begin{aligned} G^t(k,\omega) &= \frac{1-f(\epsilon_k)}{\omega-\epsilon_k+i\eta} + \frac{f(\epsilon_k)}{\omega-\epsilon_k-i\eta} \\ &= P\frac{1}{\omega-\epsilon_k} - i\pi\delta(\omega-\epsilon_k)(1-2f(\epsilon_k)) \end{aligned}$$

from equation (9) we get,

$$\Sigma^t(k,\omega) = \frac{n_iV_0^2}{V}\sum_p\left\{P\frac{1}{\omega-\epsilon_p} - i\pi\delta(\omega-\epsilon_p)(1-2f(\epsilon_p)\right\}$$

We define density of states as,

$$\rho(\epsilon) = \frac{1}{V}\sum_p\delta(\epsilon-\epsilon_p) \tag{10}$$

then, the above equation can be rewritten as,

$$\Sigma^t(k,\omega) = n_i V_0^2 \left\{ P \int_{-\infty}^{\infty} \frac{\rho(\epsilon) d\epsilon}{\omega - \epsilon} - i\pi\rho(\omega)(1 - 2f(\omega)) \right\} \tag{11}$$

Evaluation of the remaining self energies from equation (9) is on similar lines, and we get,

$$\begin{aligned} \Sigma^{\tilde{t}}(k,\omega) &= -n_i V_0^2 \left\{ P \int_{-\infty}^{\infty} \frac{\rho(\epsilon) d\epsilon}{\omega - \epsilon} + i\pi\rho(\omega)(1 - 2f(\omega)) \right\} \\ \Sigma^{>}(k,\omega) &= -i(1 - f(\omega)) 2\pi\rho(\omega) n_i V_0^2 \\ \Sigma^{<}(k,\omega) &= i f(\omega) 2\pi\rho(\omega) n_i V_0^2 \end{aligned} \tag{12}$$

And from the definition of Σ^R and Σ^A we get,

$$\begin{aligned} \Sigma^R &= n_i V_0^2 \left\{ P \int_{-\infty}^{\infty} \frac{\rho(\epsilon) d\epsilon}{\omega - \epsilon} - i\pi\rho(\omega) \right\} \\ \Sigma^A &= n_i V_0^2 \left\{ P \int_{-\infty}^{\infty} \frac{\rho(\epsilon) d\epsilon}{\omega - \epsilon} + i\pi\rho(\omega) \right\} \end{aligned} \tag{13}$$

Also

$$\Sigma^K = -i(1 - 2f(\omega)) 2\pi\rho(\omega) n_i V_0^2 \tag{14}$$