Many Body Theory Problem Set 1

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In the following problems, I use a unit system so that $\hbar = 1$ and $e = 1$.

Problem One

a. The equation of motion is

$$\frac{da(0)}{dt} = -i[a, H]$$

Substitute

$$a = \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{m\omega}}\hat{p} - i\sqrt{m\omega}\hat{x}\right]$$

$$a^\dagger = \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{m\omega}}\hat{p} + i\sqrt{m\omega}\hat{x}\right]$$

into

$$H = \omega(a^\dagger a + \frac{1}{2})$$

to get

$$H = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{x}^2$$

using $[\hat{x}_j, \hat{p}_k] = i\delta_{jk}$. This shows that that a and $a^\dagger$ can be considered as an operator factorization of H. Note that these so called creation and annihilation operators satisfy the following commutation relationship,

$$[a, a^\dagger] = 1$$

Write out the commutator explicitly and commute the a and $a^\dagger$ in the first term.

$$\begin{aligned}\frac{da(0)}{dt} &= -i\omega(aa^\dagger a + \frac{1}{2}a - a^\dagger aa - \frac{1}{2}a) \\ &= -i\omega(a^\dagger aa - a^\dagger aa + a) = -i\omega a\end{aligned}$$

Upon integration

$$a(t) = a(0)e^{-i\omega t}$$

b. Given

$$a^\dagger(t) = a^\dagger(0)e^{i\omega t}$$

And from part a

$$a^\dagger(0) = \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{m\omega}}\hat{p}(0) + i\sqrt{m\omega}\hat{x}(0)\right]$$

so

$$\begin{aligned}a(t) &= \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{m\omega}}\hat{p}(0) - i\sqrt{m\omega}\hat{x}(0)\right]e^{-i\omega t} \\ &= \left[\frac{1}{\sqrt{2m\omega}}\hat{p}(t) - i\sqrt{\frac{1}{2}m\omega}\hat{x}(t)\right]\end{aligned}$$

$$\begin{aligned}a^\dagger(t) &= \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{m\omega}}\hat{p}(0) + i\sqrt{m\omega}\hat{x}(0)\right]e^{i\omega t} \\ &= \left[\frac{1}{\sqrt{2m\omega}}\hat{p}(t) + i\sqrt{\frac{1}{2}m\omega}\hat{x}(t)\right]\end{aligned}$$

Add these

$$a(t) + a^\dagger(t) = \sqrt{\frac{2}{m\omega}}\hat{p}(t) = \frac{1}{\sqrt{2m\omega}}\hat{p}(0)\left[e^{-i\omega t} + e^{i\omega t}\right] + i\sqrt{\frac{m\omega}{2}}\hat{x}(0)\left[e^{i\omega t} - e^{-i\omega t}\right]$$

Solve for $\hat{p}$

$$\hat{p}(t) = \hat{p}(0)\cos\omega t - m\omega\hat{x}(0)\sin\omega t$$

To get $\hat{x}$, subtract

$$a^\dagger(t) - a(t) = \sqrt{2m\omega}\hat{x}(t) = \frac{1}{\sqrt{2m\omega}}\hat{p}(0)\left[e^{i\omega t} - e^{-i\omega t}\right] + i\sqrt{\frac{m\omega}{2}}\hat{x}(0)\left[e^{i\omega t} + e^{-i\omega t}\right]$$

Solve for $\hat{x}$

$$\hat{x}(t) = \hat{x}(0)\cos\omega t + \frac{1}{m\omega}\hat{p}(0)\sin\omega t$$

c. We know that $\langle q\rangle$ and $\langle p\rangle$ obey the Erenfest theorem and we can write:

$$\begin{aligned}\frac{\langle q\rangle}{dt} &= \frac{\langle p\rangle}{m}\\ \frac{\langle p\rangle}{dt} &= -\langle V'\rangle.\end{aligned}$$

And they are equivalent to classical equations when $\langle V'\rangle = \langle V'_{cl}\rangle$. If dispersion $(\Delta q)^2 = \langle q^2\rangle - \langle q\rangle^2$ of wave packet is small then we can write:

$$V'(q) = V'_{cl}(\langle q\rangle) + (q - \langle q\rangle)V''_{cl} + (q - \langle q\rangle)^2 V'''_{cl} + ...$$

We see that $V' = V'_{cl}$ if all term higher that V'' are equal to zero. Harmonic oscillator potential has quadratic form and hence quantum and classical equation of motion look similar.

Problem Two

$$\hat{S}_x = \frac{1}{2}\left(c^\dagger_\uparrow c_\downarrow + c^\dagger_\downarrow c_\uparrow\right)$$

$$\hat{S}_y = -\frac{1}{2}i\left(c^\dagger_\uparrow c_\downarrow - c^\dagger_\downarrow c_\uparrow\right)$$

$$\hat{S}_z = \frac{1}{2}\left(c^\dagger_\uparrow c_\uparrow - c^\dagger_\downarrow c_\downarrow\right)$$

Check commutators

$$[\hat{S}_x, \hat{S}_y] = -\frac{i}{4}\left[c_\uparrow^\dagger c_\downarrow c_\uparrow^\dagger c_\downarrow + c_\downarrow^\dagger c_\uparrow c_\uparrow^\dagger c_\downarrow - c_\downarrow^\dagger c_\uparrow c_\downarrow^\dagger c_\uparrow - c_\uparrow^\dagger c_\downarrow c_\downarrow^\dagger c_\uparrow\right]$$
$$+\frac{i}{4}\left[c_\uparrow^\dagger c_\downarrow c_\uparrow^\dagger c_\downarrow + c_\uparrow^\dagger c_\downarrow c_\downarrow^\dagger c_\uparrow - c_\downarrow^\dagger c_\uparrow c_\downarrow^\dagger c_\uparrow - c_\downarrow^\dagger c_\uparrow c_\uparrow^\dagger c_\downarrow\right]$$

Use

$$\{c_\sigma, c_{\sigma'}^\dagger\} = \delta_{\sigma\sigma'}$$

Normal order. Because $c_\sigma c_\sigma = 0$ and $c_\sigma^\dagger c_\sigma^\dagger = 0$, several terms vanish.

$$[\hat{S}_x, \hat{S}_y] = -\frac{i}{4}\left[-c_\downarrow^\dagger c_\uparrow^\dagger c_\uparrow c_\downarrow + c_\downarrow^\dagger c_\downarrow + c_\uparrow^\dagger c_\downarrow^\dagger c_\downarrow c_\uparrow - c_\uparrow^\dagger c_\uparrow + c_\uparrow^\dagger c_\downarrow^\dagger c_\downarrow c_\uparrow + c_\downarrow^\dagger c_\downarrow - c_\downarrow^\dagger c_\uparrow^\dagger c_\uparrow c_\downarrow - c_\uparrow^\dagger c_\uparrow\right]$$
$$= \frac{i}{2}\left[c_\uparrow^\dagger c_\uparrow - c_\downarrow^\dagger c_\downarrow\right] = i\hat{S}_z$$

$$[\hat{S}_z, \hat{S}_y] = -\frac{i}{4}\left[c_\uparrow^\dagger c_\uparrow c_\uparrow^\dagger c_\downarrow + c_\downarrow^\dagger c_\downarrow c_\downarrow^\dagger c_\uparrow - c_\downarrow^\dagger c_\downarrow c_\uparrow^\dagger c_\downarrow - c_\uparrow^\dagger c_\uparrow c_\downarrow^\dagger c_\uparrow\right]$$
$$+\frac{i}{4}\left[c_\uparrow^\dagger c_\downarrow c_\uparrow^\dagger c_\uparrow + c_\downarrow^\dagger c_\uparrow c_\downarrow^\dagger c_\downarrow - c_\uparrow^\dagger c_\downarrow c_\downarrow^\dagger c_\downarrow - c_\downarrow^\dagger c_\uparrow c_\uparrow^\dagger c_\uparrow\right]$$
$$= -\frac{i}{4}\left[-c_\uparrow^\dagger c_\uparrow^\dagger c_\uparrow c_\downarrow + c_\uparrow^\dagger c_\downarrow - c_\downarrow^\dagger c_\downarrow^\dagger c_\downarrow c_\uparrow + c_\downarrow^\dagger c_\uparrow + c_\downarrow^\dagger c_\uparrow^\dagger c_\downarrow c_\downarrow + c_\uparrow^\dagger c_\downarrow^\dagger c_\uparrow c_\uparrow\right]$$
$$-\frac{i}{4}\left[+c_\uparrow^\dagger c_\uparrow^\dagger c_\downarrow c_\uparrow + c_\downarrow^\dagger c_\downarrow^\dagger c_\uparrow c_\downarrow - c_\uparrow^\dagger c_\downarrow^\dagger c_\downarrow c_\downarrow + c_\uparrow^\dagger c_\downarrow - c_\downarrow^\dagger c_\uparrow^\dagger c_\uparrow c_\uparrow + c_\downarrow^\dagger c_\uparrow\right]$$
$$= -\frac{1}{2}i\left(c_\uparrow^\dagger c_\downarrow + c_\downarrow^\dagger c_\uparrow\right) = -i\hat{S}_x$$

$$[\hat{S}_z, \hat{S}_x] = \frac{1}{4}\left[c_\uparrow^\dagger c_\uparrow c_\uparrow^\dagger c_\downarrow - c_\downarrow^\dagger c_\downarrow c_\uparrow^\dagger c_\downarrow - c_\downarrow^\dagger c_\downarrow c_\downarrow^\dagger c_\uparrow + c_\uparrow^\dagger c_\uparrow c_\downarrow^\dagger c_\uparrow\right]$$
$$-\frac{1}{4}\left[c_\uparrow^\dagger c_\downarrow c_\uparrow^\dagger c_\uparrow - c_\uparrow^\dagger c_\downarrow c_\downarrow^\dagger c_\downarrow - c_\downarrow^\dagger c_\uparrow c_\downarrow^\dagger c_\downarrow + c_\downarrow^\dagger c_\uparrow c_\uparrow^\dagger c_\uparrow\right]$$
$$= \frac{1}{4}\left[-c_\uparrow^\dagger c_\uparrow^\dagger c_\uparrow c_\downarrow + c_\uparrow^\dagger c_\downarrow + c_\downarrow^\dagger c_\uparrow^\dagger c_\downarrow c_\downarrow + c_\downarrow^\dagger c_\downarrow^\dagger c_\downarrow c_\uparrow - c_\downarrow^\dagger c_\uparrow - c_\uparrow^\dagger c_\downarrow^\dagger c_\uparrow c_\uparrow\right]$$
$$+\frac{1}{4}\left[+c_\uparrow^\dagger c_\uparrow^\dagger c_\downarrow c_\uparrow - c_\uparrow^\dagger c_\downarrow^\dagger c_\downarrow c_\downarrow + c_\uparrow^\dagger c_\downarrow - c_\downarrow^\dagger c_\downarrow^\dagger c_\downarrow c_\downarrow + c_\downarrow^\dagger c_\uparrow^\dagger c_\uparrow c_\uparrow - c_\downarrow^\dagger c_\uparrow\right]$$
$$= \frac{1}{2}\left(c_\uparrow^\dagger c_\downarrow - c_\downarrow^\dagger c_\uparrow\right) = i(-\frac{i}{2})\left(c_\uparrow^\dagger c_\downarrow - c_\downarrow^\dagger c_\uparrow\right) = i\hat{S}_y$$

Problem Three

We start with

$$H = \sum_{i,j} a_i^\dagger a_j \langle i|\frac{\hat{p}^2}{2m}|j\rangle + \frac{1}{2}\sum_{i,j,k,n} a_i^\dagger a_j^\dagger a_k a_n \langle i,j|\frac{A}{4\pi}\frac{1}{|\mathbf{r}-\mathbf{r}'|}e^{-\lambda|\mathbf{r}-\mathbf{r}'|}|k,n\rangle$$

Insert the completeness relationship once in the first term, four times in the next term. Rewrite the result equation in terms of wave-functions.

$$H = -\sum_{i,j}\int d^3\mathbf{r}\ \frac{1}{2m} a_i^\dagger a_j \phi_i^*(\mathbf{r})\nabla^2\phi_j(\mathbf{r})$$

$$+\frac{A}{8\pi}\sum_{i,j,k,n} a_i^\dagger a_j^\dagger a_k a_n \int d^3\mathbf{r}\ d^3\mathbf{r}'\ \frac{1}{|\mathbf{r}-\mathbf{r}'|}e^{-\lambda|\mathbf{r}-\mathbf{r}'|}\phi_i^*(\mathbf{r})\phi_j^*(\mathbf{r}')\phi_k(\mathbf{r}')\phi_n(\mathbf{r})$$

Use the definitions of

$$\psi(\mathbf{r}) = \sum_j \phi_j(\mathbf{r}) a_j$$

$$\psi^\dagger(\mathbf{r}) = \sum_i \phi_i^*(\mathbf{r}) a_i^\dagger$$

to do the sums over the i,j, etc. We get

$$H = -\int d^3\mathbf{r}\ \frac{1}{2m}\psi^\dagger(\mathbf{r})\nabla^2\psi(\mathbf{r}) + \frac{A}{8\pi}\int d^3\mathbf{r}\, d^3\mathbf{r}'\ \psi^\dagger(\mathbf{r})\psi^\dagger(\mathbf{r}')\frac{1}{|\mathbf{r}-\mathbf{r}'|}e^{-\lambda|\mathbf{r}-\mathbf{r}'|}\psi(\mathbf{r}')\psi(\mathbf{r})$$

which is what we wanted.

b. Fourier transform everything. The following will be useful:

$$\psi(\mathbf{r}) = \frac{1}{(2\pi)^3}\int d^3\mathbf{k}\ c_\mathbf{k}\ e^{i\mathbf{k}\cdot\mathbf{r}}$$

$$V(\mathbf{r}-\mathbf{r}') = \frac{1}{(2\pi)^3}\int d^3\mathbf{q}\ \frac{A}{q^2+\lambda^2}e^{i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')}$$

$$\delta(\mathbf{k}_2-\mathbf{k}_1) = \frac{1}{(2\pi)^3}\int d^3\mathbf{r}\ e^{i\mathbf{r}\cdot(\mathbf{k}_2-\mathbf{k}_1)}$$

Plug these into the result from part a.

$$H = \frac{1}{(2\pi)^6}\int d^3\mathbf{r}\; d^3\mathbf{k}_1\; d^3\mathbf{k}_2\; \frac{1}{2m}k_2^2 c^\dagger_{\mathbf{k}_1} c_{\mathbf{k}_2} e^{i(\mathbf{k}_2-\mathbf{k}_1)\cdot\mathbf{r}}$$
$$+\frac{A}{16\pi^3}\frac{1}{(2\pi)^6}\int d^3\mathbf{r}\; d^3\mathbf{r}'\; d^3\mathbf{k}_1\; d^3\mathbf{k}_2\; d^3\mathbf{k}\; d^3\mathbf{k}'\; d^3\mathbf{q}$$
$$\frac{1}{q^2+\lambda^2} c^\dagger_{\mathbf{k}_1} c^\dagger_{\mathbf{k}_2} c_{\mathbf{k}'} c_{\mathbf{k}} e^{i(\mathbf{k}_1+\mathbf{q}-\mathbf{k})\cdot\mathbf{r}} e^{i(\mathbf{k}_2-\mathbf{q}-\mathbf{k}')\cdot\mathbf{r}'}$$

Pick out delta functions and do integrations over them.

$$H = \frac{1}{(2\pi)^3}\int d^3k\; \frac{1}{2m}k^2 c^\dagger_{\mathbf{k}} c_{\mathbf{k}}$$
$$+\frac{A}{16\pi^3}\int d^3k\; d^3k'\; d^3q\; \frac{1}{q^2+\lambda^2} c^\dagger_{\mathbf{k}-\mathbf{q}} c^\dagger_{\mathbf{k}'+\mathbf{q}} c_{\mathbf{k}'} c_{\mathbf{k}}$$

Problem Four

Show

$$e^{-a(n_\uparrow-\frac{1}{2})(n_\downarrow-\frac{1}{2})} = \frac{1}{2}e^{-\frac{a}{4}}\sum_{\sigma=\pm} e^{a\sigma(n_\uparrow-n_\downarrow)}$$

To prove, I'll manipulate each side into suggestive forms and then show that both sides are the same. First, consider the left hand side of the equality.

$$\text{l.h.s.} = e^{-a(n_\uparrow-\frac{1}{2})(n_\downarrow-\frac{1}{2})} = e^{-\frac{a}{4}+\frac{1}{2}a\left(n_\uparrow+n_\downarrow\right)-an_\uparrow n_\downarrow}$$

$$\text{r.h.s.} = \frac{1}{2}e^{-\frac{a}{4}}\sum_{\sigma=\pm} e^{a\sigma(n_\uparrow-n_\downarrow)}$$
$$= e^{-\frac{a}{4}}\cosh\left[a|n_\uparrow-n_\downarrow|\right]$$

I put the absolute value sign to remind us that $\cosh(x) = \cosh(-x)$.
For fermions, $n_\sigma = 0$ or 1.
If $n_\uparrow = 1$ and $n_\downarrow = 1$, then $|n_\uparrow - n_\downarrow| = 0$
If $n_\uparrow = 0$ and $n_\downarrow = 0$, then $|n_\uparrow - n_\downarrow| = 0$
If $n_\uparrow = 1$ and $n_\downarrow = 0$, then $|n_\uparrow - n_\downarrow| = 1$
If $n_\uparrow = 0$ and $n_\downarrow = 1$, then $|n_\uparrow - n_\downarrow| = 1$
In the first two cases, the hyperbolic cosine term, $\cosh(a|n_\uparrow - n_\downarrow|)$, gives one.

In the latter case, we can use the relation $\cosh \pm a = e^{\frac{1}{2}a}$ and rewrite the *r.h.s.* Equate the *r.h.s.* to the *l.h.s* and factor out common multiples of $e^{-\frac{a}{4}}$. Take the log of both sides. We'll be left with

$$-an_\uparrow n_\downarrow + \frac{1}{2}a(n_\uparrow + n_\downarrow) = \frac{1}{2}a|n_\uparrow - n_\downarrow|$$

Finally, I show that for all pairing of $n_\sigma = 0, 1$ this equality will be satisfied.
If $n_\uparrow = 1$ and $n_\downarrow = 1$, then $-a + a = 0$
If $n_\uparrow = 0$ and $n_\downarrow = 0$, then $0 + 0 = 0$
If $n_\uparrow = 1$ and $n_\downarrow = 0$, then $0 + \frac{1}{2}a = \frac{1}{2}a$
If $n_\uparrow = 0$ and $n_\downarrow = 1$, then $0 + \frac{1}{2}a = \frac{1}{2}a$
so our relation is true for fermions. All of this could have been rewritten in matrix notation.

Problem Five

$$H_0 = \sum_\alpha \epsilon_\alpha a_\alpha^\dagger a_\alpha$$

a. I use γ instead of what's on the homework assignment so that I can keep the labels straight. β has a negative sign so that the partition function is bounded.

$$\text{Trace}\left[e^{-\beta H_0} a_\gamma^\dagger a_\gamma\right] = \sum_{\{n_i\}} \langle \{n_i\}| \left[e^{-\beta H_0} a_\gamma^\dagger a_\gamma\right] |\{n_i\}\rangle$$

where $|\{n_i\}\rangle$ stands for a many particle state with a set, $\{n_i\}$, of occupation numbers. $a_\gamma^\dagger a_\gamma$ acting on $|\{n_i\}\rangle$ returns 0 or 1 depending on whether there is a fermion in state γ - Alas, the notation is a bit degenerate - We can likewise get the eigenvalues for the operators in the H-potential. I will use $e^{\sum \text{stuff}} = \prod e^{\text{stuff}}$, and write

$$\text{Trace}\left[e^{-\beta H_0} a_\gamma^\dagger a_\gamma\right] = \sum_{\{n_i\}} \prod_\alpha \langle \{n_i\}| \left[e^{-\beta \epsilon_\alpha a_\alpha^\dagger a_\alpha} a_\gamma^\dagger a_\gamma\right] |\{n_i\}\rangle$$

Now, I let the density operators act on the various γ states.

$$\text{Trace}\left[e^{-\beta H_0} a_\gamma^\dagger a_\gamma\right] = \sum_{\{n_i\}} n_\gamma \prod_\alpha \left[e^{-\beta \epsilon_\alpha n_\alpha}\right]$$

Take the term which was generated by the state γ out of the product.

$$\text{Trace}\left[e^{-\beta H_0} a_\gamma^\dagger a_\gamma\right] = \sum_{\{n_i\}} n_\gamma e^{-\beta\epsilon_\gamma n_\gamma} \prod_{\alpha\neq\gamma} \left[e^{-\beta\epsilon_\alpha n_\alpha}\right]$$

Do the sum over occupation numbers $\{n_i\}$ with $n_i = 1$ or 0.

$$\text{Trace}\left[e^{-\beta H_0} a_\gamma^\dagger a_\gamma\right] = e^{-\beta\epsilon_\gamma} \prod_{\alpha\neq\gamma} \left[1 + e^{-\beta\epsilon_\alpha}\right]$$

b. Solve

$$\text{Trace}\left[e^{-\beta H_0} a_\gamma a_\gamma^\dagger\right] = -\text{Trace}\left[e^{-\beta H_0} a_\gamma^\dagger a_\gamma\right] + \text{Trace}\left[e^{-\beta H_0}\right]$$

using $\{a_\alpha, a_\alpha^\dagger\} = 1$. We know the first term on the left from part a. The second term doesn't take too much effort to get.

$$\text{Trace}\left[e^{-\beta H_0}\right] = \sum_{\{n_i\}} \prod_\alpha \langle\{n_i\}| \left(e^{-\beta\epsilon_\alpha n_\alpha}\right) |\{n_i\}\rangle = \prod_\alpha \left(1 + e^{-\beta\epsilon_\alpha}\right)$$

So

$$\begin{aligned}\text{Trace}\left[e^{-\beta H_0} a_\gamma a_\gamma^\dagger\right] &= -e^{-\beta\epsilon_\gamma} \prod_{\alpha\neq\gamma} \left[1 + e^{-\beta\epsilon_\alpha}\right] + \prod_\alpha \left(1 + e^{-\beta\epsilon_\alpha}\right) \\ &= \left[-\frac{e^{-\beta\epsilon_\gamma}}{1 + e^{-\beta\epsilon_\alpha}} + \frac{1 + e^{-\beta\epsilon_\alpha}}{1 + e^{-\beta\epsilon_\alpha}}\right] \prod_\alpha \left(1 + e^{-\beta\epsilon_\alpha}\right)\end{aligned}$$

Which can be rewritten

$$\text{Trace}\left[e^{-\beta H_0} a_\gamma a_\gamma^\dagger\right] = \frac{1}{1 + e^{-\beta\epsilon_\gamma}} \prod_\alpha \left(1 + e^{-\beta\epsilon_\alpha}\right)$$

c. Noting that

$$e^{-\beta H_0} a_\gamma^\dagger a_\gamma = -\frac{1}{\beta}\frac{\partial}{\partial\epsilon_\gamma} e^{-\beta H_0}$$

We can find

$$\text{Trace}\left[e^{-\beta H_0} a_\gamma^\dagger a_\gamma\right] = -\frac{1}{\beta}\frac{\partial}{\partial\epsilon_\gamma} \text{Trace}\left[e^{-\beta H_0}\right]$$

¿From earlier,

$$\text{Trace}\left[e^{-\beta H_0}\right] = \prod_\alpha \left[1 + e^{-\beta\epsilon_\alpha}\right]$$

whence it follows

$$\text{Trace}\left[e^{-\beta H_0} a_\gamma^\dagger a_\gamma\right] = e^{-\beta\epsilon_\gamma} \prod_{\alpha\neq\gamma} \left[1 + e^{-\beta\epsilon_\alpha}\right]$$

which is what I got in part a.

d.

$$Z = \text{Trace}\left[e^{-\beta H_0}\right] = \prod_\alpha \left[1 + e^{-\beta\epsilon_\alpha}\right]$$

and

$$\begin{aligned}\text{Trace}\left[e^{-\beta H_0} a_\gamma^\dagger a_\gamma\right] &= e^{-\beta\epsilon_\gamma} \prod_{\alpha\neq\gamma} \left[1 + e^{-\beta\epsilon_\alpha}\right] \\ &= \frac{e^{-\beta\epsilon_\gamma}}{1 + e^{-\beta\epsilon_\gamma}} \prod_\alpha \left[1 + e^{-\beta\epsilon_\alpha}\right]\end{aligned}$$

so

$$\begin{aligned}\langle n_\gamma \rangle = \text{Trace}\left[e^{-\beta H_0} a_\gamma^\dagger a_\gamma\right] / \text{Trace}\left[e^{-\beta H_0}\right] \\ = \frac{1}{1 + e^{\beta\epsilon_\gamma}}\end{aligned}$$

e. For bosons,

$$[a, a^\dagger] = 1$$

First, we get the partition function. Notice that all for each n_i, all positive integer values are allowed. The sum is a geometric series and can be done explicitly.

$$\begin{aligned}\text{Trace}\left[e^{-\beta H_0}\right] = Z &= \sum_{\{n_i\}} \prod_\alpha e^{-\beta\epsilon_\alpha n_\alpha} \\ &= \prod_\alpha \frac{1}{1 - e^{-\beta\epsilon_\alpha}}\end{aligned}$$

Then, we use a nice little trick.

$$\begin{aligned}\text{Trace}\left[e^{-\beta H_0} a_\gamma^\dagger a_\gamma\right] &= -\frac{1}{\beta}\frac{\partial}{\partial \epsilon_\gamma}\text{Trace}\left[e^{-\beta H_0}\right] \\ &= \frac{e^{-\beta\epsilon_\alpha}}{1-e^{-\beta\epsilon_\alpha}}\prod_\alpha \frac{1}{1-e^{-\beta\epsilon_\alpha}}\end{aligned}$$

$$\begin{aligned}\langle n_\gamma\rangle = \text{Trace}\left[e^{-\beta H_0} a_\gamma^\dagger a_\gamma\right]/\text{Trace}\left[e^{-\beta H_0}\right] \\ = \frac{-1}{1-e^{\beta\epsilon_\gamma}}\end{aligned}$$

When one of the energies is zero or $\beta \to 0$, the bose function diverges, and we get a condensate.

Finally, I need to find $\text{Trace}\left[e^{-\beta H_0} a_\gamma a_\gamma^\dagger\right]$. Use the commutator to shuffle terms.

$$\text{Trace}\left[e^{-\beta H_0} a_\gamma a_\gamma^\dagger\right] = \text{Trace}\left[e^{-\beta H_0} a_\gamma^\dagger a_\gamma\right] + \text{Trace}\left[e^{-\beta H_0}\right] = \langle n_\gamma\rangle Z + Z$$

f. We work with bosons.

$$\begin{aligned}[a_\gamma, H_0] = \sum_\alpha \epsilon_\alpha [a_\gamma, a_\alpha^\dagger a_\alpha] \\ = \sum_{\alpha\neq\gamma}\epsilon_\alpha[a_\gamma a_\alpha^\dagger a_\alpha - a_\alpha^\dagger a_\alpha a_\gamma] + \epsilon_\gamma[a_\gamma, a_\gamma a_\gamma^\dagger] \\ = \epsilon_\gamma a_\gamma\end{aligned}$$

$$\begin{aligned}[a_\gamma^\dagger, H_0] = \sum_\alpha \epsilon_\alpha [a_\gamma^\dagger, a_\alpha^\dagger a_\alpha] \\ = \sum_{\alpha\neq\gamma}\epsilon_\alpha[a_\gamma^\dagger a_\alpha^\dagger a_\alpha - a_\alpha^\dagger a_\alpha a_\gamma^\dagger] + \epsilon_\gamma[a_\gamma^\dagger, a_\gamma^\dagger a_\gamma] \\ = -\epsilon_\gamma a_\gamma^\dagger\end{aligned}$$

So $[H_0, a_\gamma] = -\epsilon_\gamma a_\gamma$ and $[H_0, a_\gamma^\dagger] = \epsilon_\gamma a_\gamma^\dagger$.

$$\begin{aligned}\begin{pmatrix} a_\gamma^\dagger(\tau) \\ a_\gamma(\tau)\end{pmatrix} = e^{\tau H_0}\begin{pmatrix} a_\gamma^\dagger \\ a_\gamma\end{pmatrix} e^{-\tau H_0} \\ = \begin{pmatrix} a_\gamma^\dagger\, e^{\epsilon_\gamma \tau} \\ a_\gamma\, e^{-\epsilon_\gamma\tau}\end{pmatrix} e^{\tau H_0 - \tau H_0}\end{aligned}$$

So commuting the a and $a^\dagger$s past the H, we find

$$\begin{pmatrix} a^\dagger_\gamma(\tau) \\ a_\gamma(\tau) \end{pmatrix} = \begin{pmatrix} a^\dagger_\gamma \; e^{\epsilon_\gamma \tau} \\ a_\gamma \; e^{-\epsilon_\gamma \tau} \end{pmatrix}$$