Solution to Exercise 16

We want to diagonalize the bosonic Hamiltonian

$$\mathcal{H} = \frac{1}{2}\begin{pmatrix} a^\dagger & a \end{pmatrix}\begin{pmatrix} c & b \\ b & c \end{pmatrix}\begin{pmatrix} a \\ a^\dagger \end{pmatrix}$$

by appropriate canonical transformation. Here c, b are real and $c > b$. The operators obey bosonic commutation relation $[a, a^\dagger] = 1$.

We consider a transformation of the form

$$\begin{pmatrix} a \\ a^\dagger \end{pmatrix} = \begin{pmatrix} u & v \\ v & u \end{pmatrix}\begin{pmatrix} c \\ c^\dagger \end{pmatrix}. \tag{1}$$

Inverting the above transformation we get $[c, c^\dagger] = 1/(u^2 - v^2)$. Since, the transformation has to be canonical, we get $u^2 - v^2 = 1$. This is satisfied by $u = \cosh\theta$ and $v = \sinh\theta$. Thus, the two unknown parameters have been reduced to one. We will choose this unknown θ such that the Hamiltonian is diagonal in terms of the $c, c^\dagger$ operators. From equation (1) we get

$$\begin{aligned}
\mathcal{H} &= \frac{1}{2}\begin{pmatrix} c^\dagger & c \end{pmatrix}\begin{pmatrix} u & v \\ v & u \end{pmatrix}\begin{pmatrix} c & b \\ b & c \end{pmatrix}\begin{pmatrix} u & v \\ v & u \end{pmatrix}\begin{pmatrix} c \\ c^\dagger \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} c^\dagger & c \end{pmatrix}\begin{pmatrix} c\cosh 2\theta + b\sinh 2\theta & b\cosh 2\theta + c\sinh 2\theta \\ b\cosh 2\theta + c\sinh 2\theta & c\cosh 2\theta + b\sinh 2\theta \end{pmatrix}\begin{pmatrix} c \\ c^\dagger \end{pmatrix}.
\end{aligned} \tag{2}$$

The value of θ that diagonalizes the Hamiltonian is given by $\tanh 2\theta = -b/c$. Then, $\cosh 2\theta = c/\sqrt{c^2 - b^2}$ and $\sinh 2\theta = -b/\sqrt{c^2 - b^2}$. We get,

$$\begin{aligned}
\mathcal{H} &= \frac{1}{2}\begin{pmatrix} c^\dagger & c \end{pmatrix}\begin{pmatrix} \sqrt{c^2 - b^2} & 0 \\ 0 & \sqrt{c^2 - b^2} \end{pmatrix}\begin{pmatrix} c \\ c^\dagger \end{pmatrix} \\
&= \hbar\omega\left(c^\dagger c + \frac{1}{2}\right).
\end{aligned} \tag{3}$$

Thus, the spectrum is the usual harmonic oscillator spectrum with frequency $\hbar\omega = \sqrt{c^2 - b^2}$.