ON THE SIGNIFICANCE OF NEWTONIAN COSMOLOGY

By W. H. McCrea

Abstract. The theory of newtonian cosmology is re-examined in the light of criticisms by D. Layzer. His two theorems concerning Einstein's law of gravitation substantiate the significance that was assigned to the theory by Milne and McCrea. However, in order to provide analogues of the relativistic models that are newtonian in the strictest possible sense, the theory has to be formulated in terms of a bounded system. But, since this may be arbitrarily large, the distinction between it and an unbounded system is physically scarcely significant. Any extension to an unbounded system necessitates some extension of the definition of newtonian gravitation. Layzer's criticisms make this clear and bring out the difficulty of formulating a satisfactory extension. Nevertheless the extension that he criticises in particular is self-consistent and is the appropriate one in the context.

1. The late Professor E. A. Milne and the present writer (Milne and McCrea 1934), generalizing a particular case studied by Milne (1934), showed how a "newtonian" treatment gives a precise counterpart of every model universe given by the standard equations of relativistic cosmology provided the pressure in the model is zero. This work seems to have proved useful in affording a physical understanding of the relativistic treatment. However, Milne and I did not suggest that our work eliminated the need for a relativistic treatment of the large-scale features of the universe (McCrea 1953).

2. *Layzer's theorems.* A particular consequence of the work of Milne and myself was its indication that Einstein's law of gravitation must admit an interpretation effectively the same as that of Newton's in the case of a spherically symmetric mass-distribution. Milne actually showed this in his particular case (*loc. cit.*, pp. 70–71). The precise form of the result thus indicated has been established by D. Layzer (1954) in the form of two theorems elegantly derived from an investigation by H. Bondi. It seems fair to suggest that Layzer's work vindicates the interpretation of the relativistic formulae which Milne and I gave.

3. *Layzer's criticism.* Layzer, nevertheless, criticizes the work of Milne and myself as "being incompatible with the newtonian conception of gravitation which it tries to incorporate."

Layzer has some grounds for his criticism, as will be seen in Section 7. However, it is possible slightly to modify the presentation given by Milne and me in such a way that it is proof against this type of criticism, while giving what from a physical standpoint is effectively the same result as before. I shall first give the modified presentation and then return to a discussion of Layzer's criticism.

NEWTONIAN COSMOLOGY—BOUNDED MODEL

4. *Model.* Suppose a newtonian reference-frame $\mathfrak{N}$ to be given; let O be the origin and $\mathbf{q}$ the position-vector of a point referred to $\mathfrak{N}$; t is newtonian time.

It can be shown that the following is a system behaving strictly in accordance with standard newtonian kinematics, mechanics and gravitation:

The material is a pressure-free fluid moving under its own gravitation and no other forces.

At time t the position of a particle is

$$\mathbf{q} = R\boldsymbol{\tau} \qquad (4.1)$$

where $\boldsymbol{\tau}$ is a constant vector for that particle and $R(t)$ is a function of t only, the same for all particles. The system is composed of particles for which $|\boldsymbol{\tau}| = \tau < \tau_0$ where τ_0 is a positive constant.

At time t the density of the fluid is

$$\rho = \rho_0/R^3 \qquad (4.2)$$

where ρ_0 is a positive constant.

The function $R(t)$ is any solution of the differential equation

$$2\frac{R''}{R} + \frac{R'^2 + kc^2}{R^2} = 0 \qquad (4.3)$$

and ρ is then given by

$$3\frac{R'^2 + kc^2}{R^2} = 8\pi\gamma\rho \quad (\gamma = \text{gravitational constant}). \qquad (4.4)$$

Here kc^2 is a constant of integration, written thus because (4.3), (4.4) are then the Einstein-Friedmann equations in standard form.

The system is thus characterized completely by the function $R(t)$ and the constants kc^2, τ_0. It is to be noted that τ_0 does not enter any of the equations so that, $R(t)$, kc^2 having been selected, the parameter τ_0 determining the total extent of the system may be assigned arbitrarily.

5. *Properties.* Now consider a reference-frame $\mathfrak{A}$ of which the origin A moves with a particle A of the fluid and which is non-rotating in regard

to $\mathfrak{N}$. Let $\bar{\boldsymbol{\tau}}$ be the fixed value of $\boldsymbol{\tau}$ for A. At time t let $\mathbf{a}$ be the position vector of A relative to $\mathfrak{N}$ and $\mathbf{q}$, $\mathbf{q}_A$ the position-vectors of a particle P of the fluid relative to $\mathfrak{N}$, $\mathfrak{A}$. Then we have the vector relation

$$\mathbf{q}_A = \mathbf{q} - \mathbf{a}$$

and from (4.1)

$$\mathbf{q} = R\boldsymbol{\tau}, \quad \mathbf{a} = R\bar{\boldsymbol{\tau}}.$$

Hence

$$\mathbf{q}_A = R\boldsymbol{\tau}_A \tag{5.1}$$

where

$$\boldsymbol{\tau}_A = \boldsymbol{\tau} - \bar{\boldsymbol{\tau}}. \tag{5.2}$$

Thus the motion of the fluid relative to $\mathfrak{A}$, comparing (4.1), (5.1), is the same as its motion relative to $\mathfrak{N}$. Also, of course, the density is the same when measured in either frame. The only difference between the descriptions of the system relative to the frames $\mathfrak{N}$, $\mathfrak{A}$ is that the boundary is centered on O and not on A. Thus we have constructed a system of matter in motion which has the following properties:

Every observer moving with the material sees the same motion, which is purely radial and radially symmetric with respect to himself. If he supposes that the distribution is spherically symmetric about himself and that the classical laws of motion and gravitation hold good in his own frame, then he obtains a classically correct description of the motion. This description is given by the Einstein-Friedmann equations.

This result is rigorously in accordance with classical theory and is to be regarded as the rigorous statement of the result produced by Milne and me. It is to be emphasised that the equations of motion were got using only the postulated newtonian frame $\mathfrak{N}$. The resulting motion relative to $\mathfrak{N}$, using newtonian kinematics, yields the motion relative to any other frame $\mathfrak{A}$. This frame is in fact accelerated relative to $\mathfrak{N}$ and is therefore not newtonian; the stated properties of the motion relative to $\mathfrak{A}$ are deduced and not assumed.

Layzer mentions that such a system is obtainable by classical theory but does not state its properties relative to any observer moving with the material.

6. *Newtonian cosmology*. We now consider the system as a model of the smoothed-out universe. In doing so, as the classical theory allows us to do, we take the parameter τ_0 which determines the total extent of the system to be arbitrarily large.

We can suppose any observer to have a finite range of observation. So, in particular, we can take the extent of the system to be arbitrarily large compared with this range. Therefore the proportion of observers who can observe any part of the boundary can be taken to be arbitrarily small. There are, of course, no observers outside the system since the system is taken to represent all the contents of the universe. Thus from the standpoint of observables—the "operational standpoint"—the difference between an arbitrarily large system and an unbounded system is scarcely significant.

If we adhere to the present model, it can be seen that everything that Milne and I stated or implied about all observers moving with the material applies here to all but an arbitrarily small proportion of such observers. This is the only change in the statement of our results which is required to make them apply to a model that is strictly newtonian according to every admissible test.

We see below that any attempt to extend the discussion to an unbounded system demands some extension of the meaning of newtonian. Hence we cannot claim that any unbounded system is newtonian in the unique original sense. Therefore it is best to assert that the present bounded but arbitrarily large model is the newtonian analogue of the relativistic model given by the same function $R(t)$ and the same constant kc^2.

Whatever significance, valuable or otherwise, can be claimed for the work of Milne and myself can be claimed for it when interpreted in the present sense. Opinions may differ as to the value of possessing the newtonian analogue. But there is no doubt about its mathematical existence and its strictly newtonian character.

NEWTONIAN COSMOLOGY—UNBOUNDED MODEL

7. *Definitions*. In view of the foregoing results it could now be admitted that no unbounded system can be strictly newtonian in the original sense of the term and that any allegedly newtonian treatment of such a system must therefore necessarily be open to criticism. Layzer has advanced one possible criticism. We could merely assert that the details of such criticisms are not important because they are irrelevant to the strict newtonian treatment of a bounded (but arbitrarily large) system, which is all that we need to consider. Nevertheless there is some interest in seeing how Layzer's criticism arises.

The following are the extensions of the meaning of newtonian that may be considered:

(i) We may assert as a matter of definition that an unbounded system is newtonian if it can be regarded as the limit of a bounded newtonian system.

Milne and I attempted, in effect, to deal with our system in this way. However, in our case the limit does not exist; our results hold good for any bounded system, no matter how large, but they tell us nothing about a system occupying the whole of euclidean space.

(ii) We may assert that an unbounded system is newtonian if the infinite integrals that give the components of newtonian attraction at any point are convergent. This is the definition most commonly adopted.

According to this definition, a system with uniform density throughout euclidean space is not newtonian. As it is important to recognize, this means that, if the gravitational force is to be defined in the present manner, then it does not exist in the case of uniform density. Accordingly, nothing further can be inferred about this case. In particular, we may not proceed to argue, as Layzer does, that the force must be the same at every point, and thence that it must be zero. For, in order to prove that a force takes any value, in particular the value zero, the force has to exist in the mathematical sense.

(iii) We may assert that an unbounded system is newtonian if a function φ exists satisfying the equation

$$\nabla^2\varphi = 4\pi\gamma\rho \qquad (7.1)$$

and we then define the newtonian gravitational intensity $\mathbf{F}$ by

$$\mathbf{F} = -\operatorname{grad}\varphi. \qquad (7.2)$$

Since the system is unbounded, the boundary conditions that would be required in order to render unique the solution φ of (7.1) are not provided. Nevertheless, we may be able to find well-determined systems that satisfy the equation of continuity, the equations of motion arising from (7.1), (7.2), and whatever equation of state is prescribed. According to the present definition, these systems are newtonian. If, for two such systems, the instantaneous density-distributions at some epoch are the same, but the functions φ, and consequently the fields $\mathbf{F}$ are not the same, then the states of motion at that epoch are not the same; this is how the lack of uniqueness of φ shows itself.

In the case of bounded systems, it is a fundamental feature of newtonian gravitation that the gravitational field is uniquely determined by the instantaneous density-distribution. If we regard it as an overriding requirement that this feature be preserved in any extension to unbounded systems, then the present treatment of such systems has to be disallowed.

This appears to be the main point brought out by Layzer. It is certainly important that it should be made clear; it has been largely overlooked by other writers on newtonian cosmology.

Nevertheless, it is legitimate to assert that in passing to unbounded systems we shall not expect to satisfy the requirement stated above, precisely because we shall not have the boundary conditions that ensure its satisfaction by bounded systems. All we require is the self-consistency of the treatment.

Now, as a basis for discussion, Layzer gives in outline what is actually a formulation of newtonian cosmology according to the definition now under discussion. He then objects to it because it does not satisfy the stated requirement. But this objection arises from some different definition and does not affect the self-consistency of the formulation.

In fact, the formulation can be shown to be consistent with all of its own requirements. In particular the consistency with newtonian mechanics is demonstrated by a discussion closely similar to that in section 5. The motion relative to every co-moving frame is the same, but only one such frame has to be regarded as an ideal newtonian frame; this clears up a difficulty encountered by H. Bondi (1952).

It is significant that this formulation gives for an unbounded system the same results as those of Section 4 for a bounded system. Unlike definitions (i), (ii) it does provide an extension of these results to unbounded systems.

(iv) Instead of attempting to extend the meaning of newtonian directly, we may borrow from relativity theory results such as Layzer's theorems mentioned in Section 2. This appears to be the suggestion in the last part of Layzer's paper. The purpose of retaining a newtonian form for the calculation is then merely to describe it in familiar physical terms: we no longer have a "newtonian" cosmology.

SUMMARY

8. We can conclude as follows: Strictly newtonian analogues of the relativistic cosmological